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Josephine Evans, Havva Yoldaş. On the asymptotic behaviour of a run and tumble equation for bacterial chemotaxis. 2021. hal-03186374

**HAL Id: hal-03186374**

**<https://hal.science/hal-03186374>**

Preprint submitted on 2 Apr 2021

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# On the asymptotic behaviour of a run and tumble equation for bacterial chemotaxis

Josephine Evans\*      Havva Yoldaş †

March 30, 2021

## Abstract

We study the asymptotic behaviour of the run and tumble model for bacteria movement. Experiments show that under the effect of a chemical stimulus, the movement of bacteria is a combination of a transport with a constant velocity, “run”, and a random change in the direction of the movement, “tumble”. This so-called *velocity jump process* can be described by a kinetic-transport equation. We focus on the situation for bacteria called *E. Coli* where the tumbling rate depends on a chemical stimulus but the post tumbling velocities do not.

In this paper, we show that the linear run and tumble equation converges to a unique steady state solution with an exponential rate in a weighted total variation distance in dimension  $d \geq 1$ . We provide a constructive and quantitative proof by using Harris’s Theorem from ergodic theory of Markov processes. The result is an improvement of a recent paper by Mischler and Weng [40], since we are able to remove the radial symmetry assumption on the chemoattractant concentration. We also consider a *weakly non-linear* run and tumble equation by coupling it with a nonlocal equation on the chemoattractant concentration. We construct a unique stationary solution for the weakly non-linear equation and show the exponential convergence towards it. The novelty of our paper consist in our generalisation of the spectral gap result to dimension  $d \geq 1$  under relaxed assumptions and the methods we used in the linear setting; and, all the results in the non-linear setting.

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\*Warwick Mathematics Institute, University of Warwick, Zeeman Building, Coventry CV4 7AL, United Kingdom. josephine.evans@warwick.ac.uk

†Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 blvd. du 11 novembre 1918, 69622 Villeurbanne Cedex, France. yoldas@math.univ-lyon1.fr

# 1 Introduction and main results

We consider a kinetic-transport equation which describes the movement of biological microorganisms biased towards a chemoattractant. The model is called the *run and tumble* equation and introduced in [1, 50] based on some experimental observations [5] on the chemotaxis of the bacteria called *E. coli* towards amino-acids. The equation is given by

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathcal{V}} (T(t, x, v, v') f(t, x, v') - T(t, x, v', v) f(t, x, v)) dv', \quad t \geq 0, x \in \mathbb{R}^d, v \in \mathcal{V}. \quad (1)$$

where  $f := f(t, x, v) \geq 0$  is the density distribution of microorganisms at time  $t \geq 0$  at a position  $x \in \mathbb{R}^d$ , moving with a velocity  $v \in \mathcal{V} \subseteq \mathbb{R}^d$ . We take  $\mathcal{V} = B(0, V_0)$ , a centered ball with unit volume so that  $|\mathcal{V}| = 1$ . Microorganisms perform a biased movement along the gradient of the chemoattractant with a constant speed and they change their orientation at random times towards the regions where the chemoattractant concentration is higher. Experiments show that the duration of a run is longer than of a tumble in general. When the microorganisms move towards a favourable direction, the duration of the runs get even longer. On the other hand, if they are away from the regions of high chemoattractant density, the number of jumps increase and the run times get shorter. The underlying process is also called as the *velocity jump process*.

The tumbling frequency  $T$  describes the change in velocity from  $v$  to  $v'$  and we assume that it can be written as

$$T(t, x, v, v') := T(m, v, v') = \lambda(m)K(v, v'), \quad (2)$$

where  $\lambda : \mathbb{R} \rightarrow [0, \infty)$  is the tumbling rate and  $m$  is the derivative of the external signal  $M$  along the direction of  $v'$  and given by

$$m = v' \cdot \nabla_x M, \quad (3)$$

where  $M$  depends on the chemoattractant concentration  $S$  via

$$M = m_0 + \log S, \quad m_0 > 0, \quad (4)$$

where  $m_0$  represents the external signal in the absence of a chemical stimulus. In (2), the turning kernel  $K$  is a probability distribution on  $\mathcal{V}$  and gives the probability of moving from velocity  $v$  to velocity  $v'$  so that it satisfies

$$\int_{\mathcal{V}} K(v, v') dv' = 1.$$

More specific assumptions on  $K$  and  $\lambda$  will be given in Section 1.1 (see Hypotheses 1 and 2).

We remark that in the physically relevant case,  $m = \partial_t M + v' \cdot \nabla_x M$  as it represent the change in  $M$  bacteria experiences. However, we drop the  $\partial_t M$  term for simplicity as it is done in the previous works.

Together with the above assumptions, Equation (1) takes the form

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathcal{V}} \lambda(v' \cdot \nabla_x M(x)) K(v, v') f(t, x, v') dv' - \lambda(v \cdot \nabla_x M(x)) f(t, x, v), \quad (5)$$

for  $t \geq 0$ ,  $x \in \mathbb{R}^d$ ,  $v \in \mathcal{V}$  and it is complemented with an initial data

$$f(0, x, v) = f_0(x, v), \quad x \in \mathbb{R}^d, v \in \mathcal{V}. \quad (6)$$

Throughout the paper, we consider  $f_0$  to be a probability measure, i.e.,  $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$ , where  $\mathcal{P}(\Omega)$  denotes the space of probability measures on a space  $\Omega$ .

If the chemoattractant density  $S$  is a fixed function of  $x$ , then the problem (1)-(6) becomes linear. The linear run and tumble model was studied in numerous works including [11, 40, 42, 43]. In [11], the authors proved the existence and uniqueness of a non-trivial stationary state and exponential decay to equilibrium as  $t \rightarrow \infty$  in dimension  $d = 1$ . The technique is based on the modified entropy approach introduced in [22]. An example of a tumbling frequency satisfying the assumptions in [11] is given by

$$T(x, v, v') = 1 + \chi \operatorname{sgn}(x \cdot v), \quad \chi \in (0, 1), \quad (7)$$

where  $\chi$  is called the chemotactic sensitivity. Recently in [40], this result was extended to higher dimension  $d \geq 1$  by considering splitting techniques due to [39]. These techniques are based on using the Krein-Rutman theorem for positive semigroups which do not satisfy the necessary compactness assumption. The general form of the tumbling frequency considered in [40] is given by

$$T(x, v, v') = 1 - \chi \operatorname{sgn}(\partial_t S + v \cdot \nabla_x S), \quad \chi \in (0, 1). \quad (8)$$

In [40], the authors further assumed that the concentration of the chemoattractant  $S(x)$  is radially symmetric and decreasing in  $x$  such that  $S(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . This assumption simplifies the tumbling kernel (8) to (7) since the radial symmetry assumption reduces the problem essentially to dimension  $d = 1$ . In this paper, we are able to remove the radial symmetry assumption and obtain the exponential convergence towards a unique stationary state in dimension  $d \geq 1$ . As in our case, when the concentration of the chemoattractant  $S$  is a fixed function of  $x$  but not necessarily radially symmetric or strictly decreasing in  $|x|$ , we refer to it as the linear problem. However we remark that, in [40], the authors refer to a specific case of the run and tumble equation as the linear problem. What we call the linear equation in this paper refers to more general form of the run and tumble equation.

If the microorganisms produce a chemical agent themselves as well, then the concentration of the chemical agent  $S(t, x)$  is not a fixed, given function anymore, but it solves another equation. The physically relevant assumption in this case is that  $S$  is the solution of a Poisson type equation with a source term

$$-\Delta S + \alpha S = \rho(t, x) := \int_{\mathcal{V}} f(t, x, v) dv, \quad (9)$$

where  $\alpha \geq 0$  is the chemical degradation rate and  $\rho$  is the spatial density of microorganisms. The non-linear problem obtained by coupling (1) with (9) was first introduced in [1, 41] and further studied in [19]. In [19], the authors proved the global existence of weak solutions in dimension  $d = 3$  assuming that

$$0 \leq T(t, x, v, v') \leq C(1 + S(t, x + v) + S(t, x - v')),$$

for a nonnegative initial data  $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathcal{V})$ . Then, the global existence of weak solutions in dimensions  $d = 2$  and  $d = 3$  under similar bounds on  $T$  by the terms  $S(t, x \pm v)$  and  $|\nabla S(t, x \pm v)|$  was proved in [33]. Then in [9], the authors extended the global existence results of [19, 33] to more general tumbling frequencies by using the dispersion and Strichartz estimates of [15]. Most notably, they obtained the optimal global existence result in dimensions  $d = 3$  and  $d = 4$  for a sufficiently small initial data in the difficult case of a tumbling frequency satisfying

$$0 \leq T(t, x, v, v') \leq C (|S(t, x \pm v)| + |S(t, x \pm v')| + |\nabla S(t, x \pm v)| + |\nabla S(t, x \pm v')|),$$

where any combination of signs is allowed in the right hand side. In [8], the authors improved the global existence results of [19]. Moreover, in [7], the authors considered (1)-(9) and proved that there exists a critical mass and the solutions blow up in finite time if the initial mass is above the critical mass, whereas the solutions globally exist if the initial mass is below the critical mass. Their results

are given for a particular form of the tumbling frequency which is different than the ones previously mentioned and for a spherically symmetric initial data in dimension  $d = 2$ .

In [10], the author studied the existence of traveling wave solutions of (1) coupled with two reaction-diffusion equations for the concentration of amino-acid released by the bacteria and the concentration of nutrient consumed by the population. The author showed that under certain conditions on the parameters, travelling wave solutions exist. This analytical result complements the experimental observations and computational studies in [46, 47]. The results in [10] are however, restricted to dimension  $d = 1$ . Some numerical counterexamples for existence and uniqueness are also provided. We refer also to [8] for a detailed review of existence and blow-up results for kinetic models of chemotaxis.

In the present paper, we are concerned with the long-time behaviour of the run and tumble equation in the case that the solutions exist globally in time. Therefore we do not provide an existence result. Nevertheless, since the tumbling frequency we consider can be bounded by the necessary terms (see Hypotheses 1 and 2 in Section 1.1), the global existence result can be obtained by following the strategy in [19].

Up to the best of our knowledge, the analytical studies on the non-linear run and tumble model (1) with the Poisson coupling (9) are restricted to those we mentioned above and references therein. As for the analysis of the long-time behaviour in the case that solutions do not blow up in finite time, the results are even more scarce. This is due to the fact that the non-linear problem is more challenging to study mathematically. These challenges include proving the existence of equilibrium or non-equilibrium steady solution or solutions and convergence results.

In this paper, apart from the linear equation, we consider a nonlocal non-linear coupling as a toy model which serves as an intermediate step to treat the case with Poisson coupling (9). Let us call  $\rho(x) = \int f(x, v) dv$  the spatial marginal density of microorganisms. We consider

$$S = S_\infty(1 + \eta N * \rho), \tag{10}$$

where  $\eta > 0$  is a small constant,  $N$  is a positive, smooth function with a compact support, and  $S_\infty$  is a smooth function. We refer to problem (5) with the coupling (10) as the *weakly non-linear* run and tumble model. The reason for this coupling will be made more precise later in Section 4.

We show that there exist unique, non-trivial stationary solutions to both the *linear* and the *weakly non-linear* equations and the solutions converge to these equilibria exponentially. First, we obtain a unique stationary state for the linear equation as an application of Harris's theorem. Then we build a stationary state for the weakly non-linear equation (5)-(10) by a fixed-point argument and we show the exponential convergence by a perturbation argument. Indeed,  $S$  in (10) can be treated as a perturbation of the linear equation whenever  $N * \rho$  is decreasing or  $\eta$  is small. The explicit rates of convergence can be obtained in terms of constants given in the assumptions. Our proofs are all constructive and given in the weighted total variation distances.

## 1.1 Assumptions and main results

In this section, we list the assumptions and the main results of the paper.

The first assumption is on the turning kernel  $K$ .

**Hypothesis 1.** *We assume that the distribution of the change in the velocity due to tumbling is uniform. Therefore we consider*

$$K \equiv 1$$

*whenever appears later in the computations.*

The tumbling rate  $\lambda$  (see Equation (2)) increases when the microorganisms move far away from the regions where the chemoattractant density is high.

**Hypothesis 2.** The tumbling rate  $\lambda(m) : \mathbb{R} \rightarrow (0, \infty)$  is a function of the form

$$\lambda(m) = 1 - \chi\psi(m), \quad \chi \in (0, 1) \quad (11)$$

where  $\psi$  is a bounded, odd, increasing function and  $m\psi(m)$  is differentiable with  $\|\psi\|_\infty \leq 1$ .

Next, we assume that the chemoattractant density decreases as  $|x| \rightarrow \infty$ .

**Hypothesis 3.** We suppose that  $M(x) \rightarrow -\infty$  as  $|x| \rightarrow \infty$ ,  $|\nabla_x M(x)|$  is bounded, i.e., there exist  $R \geq 0$  and  $m_* > 0$  such that whenever  $|x| > R$  we have

$$|\nabla_x M(x)| \geq m_*.$$

**Hypothesis 4.** We suppose that  $\text{Hess}(M)(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  and  $|\text{Hess}(M)(x)|$  is bounded.

Our last assumption is the following:

**Hypothesis 5.** There exist a constant  $\tilde{\lambda} > 0$ , depending on  $\psi$  and  $\|\nabla_x M\|_\infty$ , and an integer  $k > 0$ , depending on  $\psi$ , such that

$$\int_{\mathcal{V}} \psi(v' \cdot \nabla_x M(x)) v' \cdot \nabla_x M(x) \, dv' \geq \tilde{\lambda}(\psi, \|\nabla_x M\|_\infty) |\nabla_x M(x)|^k. \quad (12)$$

In order to explain where Hypothesis 5 comes from and justify its use we briefly prove it in two cases.

**Lemma 1.1.** If  $\psi(z) = \text{sgn}(z)$  then Hypothesis 5 holds with  $k = 1$  and

$$\tilde{\lambda} = \int_{-V_0}^{V_0} |v_1| (V_0^2 - v_1^2)^{(d-1)/2} \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2 + 1)} \, dv_1.$$

If  $\psi$  is differentiable with  $\psi'(0) > 0$  then Hypothesis 5 holds with  $k = 2$ , and  $\tilde{\lambda}$  depends on the exact form of  $\psi$ .

*Proof.* We look at

$$\int_{\mathcal{V}} \psi(v' \cdot \nabla_x M(x)) v' \cdot \nabla_x M(x) \, dv'.$$

Since  $\mathcal{V}$  is a ball of radius  $V_0$ , by rotation we obtain

$$\int_{\mathcal{V}} \psi(v' \cdot \nabla_x M(x)) v' \cdot \nabla_x M(x) \, dv' = \int_{\mathcal{V}} \psi(v_1 |\nabla_x M(x)|) v_1 |\nabla_x M(x)| \mathbb{1}_{\{v_2^2 + \dots + v_d^2 \leq V_0^2 - v_1^2\}} \, dv_1.$$

Integrating out  $v_2, \dots, v_d$  gives

$$\int_{-V_0}^{V_0} \psi(v_1 |\nabla_x M(x)|) v_1 |\nabla_x M(x)| (V_0^2 - v_1^2)^{(d-1)/2} \frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2 + 1)} \, dv_1. \quad (13)$$

We can bound (13) below by

$$\frac{\pi^{(d-1)/2}}{\Gamma((d-1)/2 + 2)} (V_0/2)^{d-1} \int_{-V_0/2}^{V_0/2} \psi(v_1 |\nabla_x M(x)|) v_1 |\nabla_x M(x)| \, dv_1.$$

From this point we extract the first result on  $\psi(z) = \text{sgn}(z)$ .

For the case where  $\psi$  is differentiable, we continue using the fact that  $\frac{\pi^{d/2}}{\Gamma(d/2+1)}V_0^d = 1$  and changing variables from  $v_1$  to  $y = v_1|\nabla_x M|$ , then the above bound is equal to

$$\frac{1}{2^{d-1}\sqrt{\pi}} \frac{\Gamma(d/2+1)}{\Gamma((d-1)/2+1)} \frac{1}{|\nabla_x M|V_0} \int_{-V_0|\nabla_x M|/2}^{V_0|\nabla_x M|/2} \psi(y)y \, dy.$$

Note that  $\psi(y)y$  is a positive, even function which is 0 at  $y = 0$ . We have an average of  $\psi(y)y$  over  $-V_0|\nabla_x M|, V_0|\nabla_x M|$  and it approaches to 0 as  $|\nabla_x M(x)| \rightarrow 0$ . Since  $\psi$  is differentiable then  $y\psi(y) \approx \psi'(0)y^2$  when  $y$  is small so as  $|\nabla_x M| \rightarrow 0$  we obtain

$$\frac{1}{2^{d-1}\sqrt{\pi}} \frac{\Gamma(d/2+1)}{\Gamma((d-1)/2+1)} \frac{1}{|\nabla_x M|V_0} \int_{-V_0|\nabla_x M|/2}^{V_0|\nabla_x M|/2} \psi(y)y \, dy \approx \frac{1}{2^{d-1}\sqrt{\pi}} \frac{\Gamma(d/2+1)}{\Gamma((d-1)/2+1)} \psi'(0) \frac{1}{12} V_0^2.$$

This approximation only holds true as  $|\nabla_x M|$  goes to 0, but since  $|\nabla_x M|$  is a bounded function, and  $\frac{1}{2^{d-1}\sqrt{\pi}} \frac{\Gamma(d/2+1)}{\Gamma((d-1)/2+1)} \frac{1}{|\nabla_x M|V_0} \int_{-V_0|\nabla_x M|/2}^{V_0|\nabla_x M|/2} \psi(y)y \, dy$  is a continuous function of  $|\nabla_x M|$  we have the result.  $\square$

*Remark 1.2.* Hypotheses 3, 4 and 5 can be verified also in the case where Poisson coupling (9) is considered. The solution of  $-\Delta W_y(x) + \alpha W_y(x) = \delta_x$  is called *Yukawa potential* and given by the Green's function

$$W_y(x) = \int_0^\infty \frac{1}{(4\pi y)^{d/2}} \exp\left(-\frac{|x|^2}{4y} - \alpha y\right) dy,$$

and

$$-\frac{\log W_y(x)}{\sqrt{\alpha}|x|} \rightarrow 1 \text{ as } |x| \rightarrow \infty.$$

for dimension  $d \geq 1$  (see [36], Theorem 6.23). We can see that  $|\nabla_x M(x)|$  and  $|\text{Hess}(M)(x)|$  are bounded and  $\text{Hess}(M)(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , where  $M(x) = \log W_y(x)$ .

Moreover the solution of  $-\Delta S(x) + \alpha S(x) = \rho$  is given by

$$S(x) := W_y * \rho = \int_{\mathbb{R}^d} W_y(x)\rho(t, y) \, dy.$$

This case requires extra assumptions on  $\rho$  in order to verify Hypotheses 3, 4 and 5. Since we do not deal with the Poisson coupling in this paper, we skip further details.

**Main results** We state the main results of the paper below.

**Theorem 1.3** (The linear equation). *Suppose that  $t \mapsto f_t$  is the solution of Equation (5) with initial data  $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$ . We suppose that Hypotheses 1-5 are satisfied. Then there exist positive constants  $C, \sigma$  (independent of  $f_0$ ) such that*

$$\|f_t - f_\infty\|_* \leq C e^{-\sigma t} \|f_0 - f_\infty\|_*, \quad (14)$$

where  $f_\infty$  is the unique steady state solution to Equation (5). The norm  $\|\cdot\|_*$  is the weighted total variation norm defined by

$$\|\mu\|_* := \int_{\mathbb{R}^d} \int_{\mathcal{V}} (1 - \gamma v \cdot \nabla_x M(x) - \beta \gamma \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x)) e^{-\gamma M(x)} |\mu| \, dv \, dx, \quad (15)$$

where  $\gamma, \beta > 0$  are constants which can be computed explicitly. Furthermore, if there exist positive constants  $C_1, C_2$ , and  $\alpha$  such that

$$C_1 - \alpha \langle x \rangle \leq M(x) := \log(S(x)) \leq C_2 - \alpha \langle x \rangle,$$

then using equivalence of norms we can show a contraction as in (14) (with different constants  $C$  and  $\sigma$ ) in the norm

$$\|\mu\|_{**} := \int_{\mathbb{R}^d} \int_{\mathcal{V}} e^{\delta \langle x \rangle} |\mu| \, dv \, dx, \quad (16)$$

where  $\delta$  is a constant which is small enough depending on  $M$  and  $\langle x \rangle = \sqrt{1 + |x|^2}$ .

**Theorem 1.4** (The weakly non-linear equation). *Suppose that  $t \mapsto f_t$  is the solution of Equation (5) with the weakly non-linear coupling (10) where we suppose that  $N$  is a positive, smooth function with a compact support,  $\eta > 0$  is a constant, and  $S_\infty$  is a smooth function satisfying for some  $C_1, C_2, \alpha > 0$  that*

$$C_1 - \alpha \langle x \rangle \leq M_\infty(x) := \log(S_\infty(x)) \leq C_2 - \alpha \langle x \rangle,$$

where  $\langle x \rangle = \sqrt{1 + x^2}$ . We suppose that Hypotheses 1-5 are satisfied and that  $\psi$  is a Lipschitz function. Then there exists some constant  $\tilde{C}$  depending on  $C_1, C_2$ , and  $\alpha$  such that if  $\eta < \tilde{C}$  then there exists a unique steady state solution to Equation (5) with the weakly non-linear coupling (10). Suppose further that, any initial data  $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$  satisfying

$$\|f_0\|_{**} < \frac{1}{4} \left( \frac{\sigma^2}{4\eta\chi V_0 D \|\psi'\|_\infty \|\nabla_x N\|_\infty} - C^* \right),$$

where  $\sigma, D$  and  $C^*$  are found in Theorem 1.3, Proposition 3.1 and Lemma 3.4 respectively. Then we have that

$$\|f_t - f_\infty\|_{**} \leq C e^{-\sigma t/2} \|f_0 - f_\infty\|_{**},$$

where  $C$  and  $\sigma$  are some positive constants, and  $\|\cdot\|_{**}$  is defined in (16).

Proofs of these theorems are given at the end of Sections 2 and 3 respectively.

**Structure of the paper** This paper is organised as follows. In Section 1.1, we listed the assumptions which are needed throughout the paper and presented the main results. In Section 1.2, we mention the novelty of our results and discuss our motivation and methodology. In Section 1.3, we revise macroscopic models for chemotaxis briefly. We perform a parabolic scaling for (5) and show that in the limit we obtain an aggregation-diffusion equation. Our convergence result in the linear case is given by Harris's theorem. In Section 2, after a brief introduction to the mathematical framework, we give the statement of Harris's theorem, and, later we show how we verify the hypotheses of Harris's theorem for the linear run and tumble equation in the subsequent two subsections. We give the proof of Theorem 1.3 at the end of Section 2. Section 3 is devoted to the weakly non-linear case with nonlocal coupling. In this section, we prove that there exists a unique stationary state solution and exponential convergence to this solution. In Section 4, we discuss our results and their connection with the non-linear case when different couplings for the chemoattractant density are considered. We also discuss future works.

## 1.2 Motivation, methodology and novelty

**Motivation** Our main motivation in this work is to show that there exist simpler and more efficient methods allowing more generalisations and building an intermediate step to deal with stronger non-linearities in the model corresponding to physically more relevant cases. We start with our motivation to study the asymptotic behaviour of the linear equation, particularly, how it differs from similar kinetic equations and requires different methodologies than those which have been used in the previous literature.

The linear equation (5) has a structure similar to several equations appearing in the kinetic theory of gasses. In particular we mention a *linear Boltzmann equation* of the form

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \left( \int_{\mathbb{R}^2} f(t, x, v') dv' \right) \mathcal{M}(v) - f(t, x, v),$$

where  $f := f(t, x, v)$  is the density distribution of particles at time  $t$  in the phase space  $(x, v)$ ,  $V(x)$  is the confining potential, and  $\mathcal{M}(v)$  is the Maxwellian velocity distribution. Long time behaviour for such equations is studied in the field of *hypoocoercivity*. We mention Villani's memoir [51] as the work which began the study of *hypoocoercivity* as a coherent behaviour common to many kinetic equations. The linear Boltzmann equation was first shown to converge to equilibrium by Hérau in [31] and also falls under the scope of the powerful general theorem in [22]. In [16], written by the authors and others, we show that Harris's theorem from Markov process theory provides an alternative way of showing convergence to equilibrium for the linear Boltzmann equation amongst other equations.

The run and tumble equation differs from the linear Boltzmann, and similar hypoocoercive equations, in two key ways. Firstly, the confinement mechanism in the linear Boltzmann is through a 'confining field'  $\nabla_x V(x)$  whereas in the run and tumble equation the confinement is induced by the bias in the tumbling rate. This more complex confinement mechanism in the run and tumble equation is considerably more difficult to deal with. The second important difference between the linear Boltzmann equation and the run and tumble equation is the nature of the steady states. The steady states for the linear Boltzmann equation are simple and explicit and properties, such as Poincaré inequalities are immediate for such states. For the run and tumble equation, existence of a steady state is a problem in and of itself. The steady states for the run and tumble equation interact in a more complex way with the tools of hypoocoercivity. A good example of this is the fact that it is a condition for the theorem in [22] that the steady state must be in the kernel of both the *transport* and *collision* operators separately. This is not possible for a steady state of the run and tumble equation, although we define the *transport* and *collision* parts of the operator. This behaviour is similar to non-equilibrium steady states in kinetic theory such as the ones discussed in [2, 12, 13, 14, 25]. Harris's theorem is well adapted to dealing with complex non-explicit steady states, and gives the existence of a steady state and the convergence to that steady state simultaneously. This fact was exploited by the first author in [26] where we used Harris's theorem to find existence of a steady state for a non-linear kinetic equation with nonequilibrium steady states. Moreover, in [16], we showed that Harris's theorem can be applied efficiently to kinetic equations with nonlocal collision operators to obtain quantitative hypoocoercivity results. In conclusion, the classical tools from hypoocoercivity are difficult to apply on the run and tumble equation but Harris's approach gives promising results.

Our motivation behind considering the weakly non-linear equation is to provide a useful intermediate step to treat the biologically more realistic couplings by means of exploring how a similar approach to ours in this paper can be applied to the fully non-linear case. This point is discussed in Section 4 in detail.

**Methodology** We obtain the spectral gap result in the linear case by applying Harris's theorem. In our case the Foster-Lyapunov condition which is necessary to use Harris's theorem is inspired from the

moment estimates in [40]. Using this type of argument to study asymptotic behaviour of biological models is a recent topic of research. One of the important recent results in this direction was [27] where the author used Doeblin’s theorem to obtain a spectral gap result for the renewal equation. In [3, 4, 17, 20], Doeblin’s and Harris’s theorems were used for showing exponential contraction in weighted total variation distances for positive conservative and/or non-conservative semigroups, with several applications in population dynamics including the growth-diffusion and the growth-fragmentation equations. In [18], the authors used Doeblin’s theorem to show exponential convergence to equilibrium for elapsed-time structured nonlocal PDEs describing the dynamics of interacting neuron populations. They considered a perturbation of the linear case and obtained exponential relaxation results in the weakly non-linear case as well. Although using perturbative techniques to study low- and high-connectivity regimes (corresponding to a weak and strong non-linearity respectively) for the networks of interacting neurons is not new; the authors in [18] presented simplified and improved results on the weakly non-linear case. The technique is promising for similar models. Their approach allows to construct a steady solution to the non-linear equations based on an explicit smallness assumption on the connectivity parameter and the uniqueness of the stationary solution is proved by a fixed point argument. The result on the asymptotic behaviour of the weakly non-linear case is treated by proving estimates on the difference between the non-linear and the linear operators of the corresponding equations which can be understood as a perturbation of the linear equation. The methodology used in the non-linear setting in this paper is close to the ideas in [18]. Our fixed point argument is applied to a function which is a composition of two functions, specifically a logarithm of a convolution of the spatial density of probability distribution of bacteria with a smooth function. In this regard, the proof in our case is more involved as it requires the use of Harris’s theorem and unlike in [18] Doeblin’s theorem does not work. Moreover, differently than [18], our argument requires additional moment estimates for the perturbation term. We carry out this by finding an appropriate Lyapunov functional in the non-linear case as well. This was not needed in [18] as the authors could work with the steady solutions of the weakly nonlinear equation explicitly.

**Novelty** The main results of this paper are stated in Section 1.1. We give a spectral gap result for the run and tumble model in the *linear* case and exponential convergence towards the steady state solution. We use Harris’s theorem to obtain this result. Moreover, we prove that there exists a unique steady state solution for the *weakly non-linear* run and tumble equation with a *nonlocal* coupling (10). We show the exponential relaxation to the unique equilibrium in the non-linear case as well. The latter result is given by a contraction mapping argument and a perturbation of the linearised case. We give our results in the space of probability measures equipped with weighted total variation distances. Our proofs are all constructive and the convergence rates are explicitly computable in terms of the parameters given in the assumptions. The novelty in the present work is twofold. First, our results in the linear setting is a general version compared to those in the literature concerning the long-time behaviour of solutions of the run and tumble model. The result is an improvement of the recent work [40] where the authors generalised the spectral gap result to dimension  $d \geq 1$ . However, their radial symmetry assumption on the fixed chemoattractant density  $S$  collapses the problem into dimension  $d = 1$ . Our results do not require this assumption to hold and generalise other assumptions made in the previous literature. For example, the form of the tumbling frequency (8) we consider is more general, including the common assumption involving the “sign function” in the linear setting. Second, our results in the non-linear setting are new. A nonlocal coupling (10) has not been considered in the literature before and there is not any explicit convergence result in the non-linear setting with any other type of non-linearity. We believe that our results on the weakly non-linear run and tumble equation are significant as they can be considered as an intermediate step towards studying the physically relevant case with Poisson coupling (9) and also shows that the argument we use is robust. Our future goal is to study the long-time behaviour of this case. We explain the details and future works in Section 4.

### 1.3 Macroscopic models for chemotaxis

In this section, we briefly mention the connection between the macroscopic and the kinetic descriptions of chemotaxis. We provide a derivation of an aggregation-diffusion equation from (5) in the parabolic scaling limit. In the macroscopic level, the bacterial chemotaxis is defined via PDEs describing the time evolution of the population density depending on mean flux of the entire population. There is a wide literature on the macroscopic models for chemotaxis dating back to Patlak [44], Keller and Segel [35]. In [35], the authors study the aggregation behaviour of a population of a cell called *D. discoideum* which performs *amoeboid movement* by changing its shape to engulf bacteria or other substances like nutrients. They obtained a system of aggregation-diffusion equations given by

$$\partial_t \rho = \nabla \cdot (D_\rho \nabla \rho - \chi \rho \nabla S), \quad (17)$$

$$\partial_t S = D_S \Delta S + g(\rho, S), \quad (18)$$

where  $\rho := \rho(t, x)$  is the cell density and  $S := S(t, x)$  is the chemoattractant concentration for  $t \geq 0$  and  $x \in \mathbb{R}^2$ . In system (17)-(18),  $D_\rho > 0$  and  $D_S > 0$  are the diffusivity of the cells and the chemoattractant respectively,  $\chi \geq 0$  is the chemotactic sensitivity. Note that  $\chi = 0$  corresponds to the absence of a chemical stimulus. Equations (17)-(18) model the dynamics of the chemoattractant density by means of the diffusion of  $S$  and the function  $g$  which describes the production, degradation and consumption of  $S$  by the cells. The system (17)-(18) is referred to as classical Patlak-Keller-Segel (PKS) model. Typically, the cell population tends to move collectively towards the regions with higher nutrient density. After consuming all the nutrient available in their environment, cells start to disperse uniformly over the space. Then, after some time, they start to aggregate and form clusters. The aggregation describes the instability observed in the population level and it is analogous with many physical problems. The significance of the PKS model comes from the fact that it allows to investigate aggregation behaviour of the population.

The existence of solutions to the PKS model is a subject of many works. As we are not concerned with the analysis of (17)-(18) in this paper we chose to skip many of them. We refer to [6, 32] and references therein for extensive reviews of recent results.

Moreover, there are several results linking the mesoscopic and microscopic descriptions of chemotaxis to the macroscopic one. In [1], the author derived Equation (17) in a general case of dimension  $d \geq 1$  from a stochastic description of a specific model of chemotaxis. We also refer to [24, 49] for the derivation of reaction- and aggregation-diffusion equations from interacting stochastic many-particle systems in a general setting. In [42], the authors developed a diffusion approximation of a kinetic-transport equation of the form (1) where  $T$  depends only on pre- and post-tumbling velocities  $v$  and  $v'$  respectively. In [43], the authors studied the limiting behaviour of the diffusion equation obtained in [42]. They showed that the classical PKS model, with a given  $S$  which is smooth enough, can be obtained formally from the limiting behaviour of a kinetic description of chemotaxis. This derivation is carried out via the drift-diffusion expansion which is based on considering that the bias in chemotaxis is a small perturbation of the unbiased part. More recently in [45], the authors derived Keller-Segel type of macroscopic equations from two classes of kinetic-transport equations, the first one being (1). In the second class of kinetic models, the tumbling frequency depends also on the intra-cellular molecular content. In [45], the derivation of the second type of model from the first one with an appropriate scaling was also carried out. For other recent results about the derivation of Keller-Segel type macroscopic equations from kinetic-transport equations we refer to [34, 48, 52] and references therein.

**The aggregation-diffusion equation** Following [42, 43] we consider a parabolic scaling for the time and space variables in Equation (5). The scaling is based on the assumption that in a unit time interval, there are many jumps but a small net displacement. Considering this type of scaling is common when

describing the asymptotic behaviour of a velocity jump process as a diffusion process. The existence of the diffusion limit of (5) is guaranteed by the positivity of the so-called *turning* or *tumbling operator*

$$\Theta f = -\lambda(m)f + \int_{\mathcal{V}} \lambda(m)K(v, v')f(t, x, v') dv'.$$

In [42] the authors analysed the diffusion limit of the run and tumble equation in dimension  $d = 1$  when both the turning rate  $\lambda$  and the turning kernel  $K$  are constants. In our case  $\lambda$  is not constant.

We call  $\tau$  and  $\xi$  the scaled time and space variables respectively. For a small  $\varepsilon > 0$ ,  $\tau$  and  $\xi$  are given by

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon t.$$

We define  $\lambda^\varepsilon(v, \xi) := \lambda(v \cdot \nabla_x M(\xi))$  assume that as  $\varepsilon \rightarrow 0$

$$\lambda^\varepsilon(v, \xi) \approx 1 - \varepsilon \chi \psi(v \cdot (\nabla_x M)(\xi)).$$

This is consistent with the form of  $\lambda$  we assumed in this paper (see Hypothesis 2).

We call  $F(\tau, v, \xi)$  the density distribution of microorganisms with the scaled variables and we have the following equation for  $F$ ,

$$\varepsilon^2 \partial_\tau F + \varepsilon v \cdot \nabla_\xi F = \int_{\mathcal{V}} \lambda^\varepsilon(v', \xi) K(v, v') F(\tau, v', \xi) dv' - \lambda^\varepsilon(v', \xi) F.$$

We define the new spatial density,

$$\rho(\tau, \xi) := \int F(\tau, v, \xi) dv.$$

We then have by formal computations in the limit  $\varepsilon \rightarrow 0$ ,

$$\partial_\tau \rho = \nabla_\xi \cdot (\nabla_\xi \rho - u_c(\xi) \rho), \tag{19}$$

where the macroscopic chemotactic velocity  $u_c$  is given by

$$u_c = \chi \int_{\mathcal{V}} v' \psi(v' \cdot (\nabla_x M)(\xi)) dv'.$$

We can also write (19) as

$$\partial_\tau \rho = \nabla \cdot (\nabla \rho + \rho \nabla U), \quad \nabla U = -u_c(\xi).$$

or equivalently as a gradient flow of  $U$

$$\partial_\tau \rho - \nabla \cdot \left( \rho \nabla \left( \frac{\delta U}{\delta \rho}(\rho) \right) \right) = 0, \tag{20}$$

where the entropy variable, i.e., the Fréchet derivative of  $U$  is defined as,

$$\frac{\delta U}{\delta \rho}(\rho) = U(\xi)' = -\chi \int_0^\xi u_c(y) dy = -\chi \int_0^\xi \int_{\mathcal{V}} v' \psi(v' \cdot (\nabla_y M)(y)) dv' dy.$$

Moreover, the following coupling for  $S$  can be considered along with (19)

$$\varepsilon \partial_t S = \Delta S - \alpha S + \rho, \tag{21}$$

where  $\alpha$  is the diffusion rate across the surface. Assuming that the chemoattractant reaches the equilibrium much faster than the density of bacteria, we take  $\varepsilon \rightarrow 0$  and (21) becomes a parabolic-elliptic equation. We can write  $S = -W * \rho$  where  $W$  is the Newtonian potential when  $\alpha = 0$  or the Yukawa potential when  $\alpha > 0$ .

- Newtonian potential for  $\alpha = 0$  is given by

$$W_n(x) = \begin{cases} \frac{1}{2\pi} \log(|x|), & d = 2, \\ \frac{1}{d(2-d)\omega_d} |x|^{2-d}, & d \neq 2, \end{cases} \quad (22)$$

where  $\omega_d$  is the volume of the unit ball. In dimension  $d = 3$ ,  $W_n(x) = -1/(4\pi|x|)$ .

- Yukawa potential for  $\alpha > 0$  for  $x \in \mathbb{R}^d \setminus \{0\}$  is given by

$$W_y(x) = \int_0^\infty \frac{1}{(4\pi y)^{d/2}} \left( -\frac{|x|^2}{y} - \alpha y \right) dy. \quad (23)$$

Note that  $W_y(x) = 1/(\sqrt{\alpha})e^{-\sqrt{\alpha}|x|}$  and  $W_y(x) = 1/(4\pi|x|)e^{-\sqrt{\alpha}|x|}$  in dimension  $d = 1$  and  $d = 3$  respectively.

## 2 Harris's Theorem

In this section, we give the statement of Harris's theorem based on [28, 29]. Harris's theorem is a probabilistic method which gives simple conditions on ergodic (long-time) behaviour of Markov processes. The original idea dates back to Doeblin [21] where he showed *mixing* of a Markov chain whose transition probabilities possess a uniform lower bound. We refer to this condition as *Doeblin condition* and explain it below. The mixing of a Markov chain refers to the time until the Markov chain reaches its stationary state distribution. In [30], Harris studied the necessary conditions for a Markov process to admit a unique stationary state or an invariant measure. Later in [23, 37, 38], this result was used for the first time to obtain quantitative convergence rates based on verifying a *minorisation condition* and a *geometric drift* or *Foster-Lyapunov condition*. In [29], the authors provided a simplified proof of Harris's theorem by using appropriate Kantorovich distances. We state the theorems below in the spirit of [28, 29].

We consider a Polish space  $\Omega$  and denote  $\Sigma$  as the  $\sigma$ -algebra of Borel subsets of  $\Omega$ . Then  $(\Omega, \Sigma)$  is a measurable space; and, endowed with any probability measure,  $\Omega$  is a Lebesgue space. We denote the space of probability measures by  $\mathcal{P}(\Omega)$ .

A natural way to construct a Markov process is via a *transition probability function*.

**Definition 2.1.** A linear, measurable function  $\mathcal{M}(x, A)$  is a transition probability function on  $(\Omega, \Sigma)$  if for every  $x$ ,  $\mathcal{M}(x, \cdot)$  is a probability measure on  $(\Omega, \Sigma)$  and  $\mathcal{M}(\cdot, A)$  is a measurable function for every  $A \in \Sigma$ .

A Markov operator  $M$  and its adjoint  $M^*$  can be defined by means of a transition probability function  $\mathcal{M}$  in the following way:

$$(M\mu)(A) = \int_{\Omega} \mathcal{M}(x, A) |\mu| dx, \quad (M^*\phi)(x) = \int_{\Omega} \phi(y) \mathcal{M}(x, dy),$$

where  $\phi : \Omega \mapsto [0, +\infty)$  a bounded measurable function.

**Definition 2.2.** A family of Markov operators  $(M_t)_{t \geq 0}$  is called a *Markov semigroup* if it satisfies the following

- i.  $M_0 = \text{Id}$  or equivalently  $\mathcal{M}_0(x, \cdot) = \delta_x$  for all  $x \in \Omega$ .
- ii. The semigroup property:  $M_{t+s} = M_t M_s$  for  $t, s \geq 0$ .

iii. For every  $\mu \in L^1$ ,  $t \mapsto M_t\mu$  is continuous.

We also note that Markov semigroups have

i. Positivity property:  $M_t \geq 0$  for any  $t \geq 0$

ii. Conservativity property:  $\langle M_t f \rangle = \langle f \rangle$  for any  $f \in \Omega$  where  $\langle f \rangle := \langle f, \mathbf{1} \rangle$ .

In our setting  $M_t\mu$  will be the solution of the partial differential equation  $f$  at time  $t$  with an initial data  $\mu$  which is a probability measure. Moreover for every  $t \geq 0$ , if  $M_t\mu = \mu$ , then the probability measure  $\mu$  is called an *invariant measure* of  $(M_t)_{t \geq 0}$  or equivalently a *steady state solution* of  $f$ .

**Theorem 2.3** (Doebelin's Theorem). *Suppose that we have a Markov semigroup  $(M_t)_{t \geq 0}$  which satisfies*

**Doebelin's condition:** *There exists a time  $T > 0$ , a probability distribution  $\nu$  and a constant  $\alpha \in (0, 1)$  such that for any  $z_0$  in the domain*

$$M_T \delta_{z_0} \geq \alpha \nu.$$

*Then for any two finite measures  $\mu_1$  and  $\mu_2$  and any integer  $n \geq 0$  we have that*

$$\|M_T^n(\mu_1 - \mu_2)\|_{\text{TV}} \leq (1 - \alpha)^n \|\mu_1 - \mu_2\|_{\text{TV}}.$$

*As a consequence, the semigroup has a unique invariant probability measure  $\mu_\infty$ , and for all probability measures  $\mu$*

$$\|M_t(\mu - \mu_\infty)\|_{\text{TV}} \leq C e^{-\sigma t} \|\mu - \mu_\infty\|_{\text{TV}}, \quad \text{for all } t \geq 0,$$

*where  $C := 1/(1 - \alpha) > 1$  and  $\sigma := -\log(1 - \alpha)/T > 0$ .*

Doebelin's condition sometimes referred as the *strong positivity condition* or *uniform minorisation condition*. It means for a Markov process that the probability of transitioning from any initial state to any other state is positive. Doebelin's theorem gives a unique stationary state for a Markov process and exponential convergence to this state once Doebelin's condition is satisfied. However, proving such a uniform positivity is often difficult. Especially when the state space of the Markov process is unbounded. Harris's theorem is an extension of Doebelin's theorem to these cases. Instead of a uniform minorisation condition, we show that Doebelin's condition is satisfied only in a given region and verify that the process will visit this region often enough. For the latter part we need to find an appropriate Lyapunov functional, i.e., verify the Foster-Lyapunov condition.

**Theorem 2.4** (Harris's Theorem). *Suppose that we have a Markov semigroup  $(M_t)_{t \geq 0}$  satisfying the following two conditions*

**Foster-Lyapunov condition:** *There exists  $\lambda > 0$ ,  $K \geq 0$ , some time  $T > 0$  and a measurable function  $\phi$  such that for all  $z$  in the domain*

$$(M_T^* \phi)(z) \leq \lambda \phi(z) + K. \tag{24}$$

**Minorisation condition:** *There exists a time  $T > 0$ , a probability distribution  $\nu$  and a constant  $\alpha \in (0, 1)$  such that for any  $z_0 \in \mathcal{C}$ ,*

$$M_T \delta_{z_0} \geq \alpha \nu, \tag{25}$$

*where  $\mathcal{C} := \{z : \phi(z) \leq R\}$ , for some  $R > 2K/(1 - \alpha)$ .*

Then there exist  $\beta > 0$  and  $\bar{\alpha} \in (0, 1)$  such that

$$\|M_T^n(\mu_1 - \mu_2)\|_{\phi, \beta} \leq \bar{\alpha} \|\mu_1 - \mu_2\|_{\phi, \beta}$$

for all nonnegative measures  $\int \mu_1 = \int \mu_2$  where the norm  $\|\cdot\|_{\phi, \beta}$  is defined by

$$\|\mu\|_{\phi, \beta} := \int (1 + \beta\phi(z))|\mu| dz.$$

Moreover, the semigroup has a unique invariant probability measure  $\mu_\infty$  and there exist  $C > 1$ ,  $\sigma > 0$  (depending on  $T, \alpha, \lambda, K, R$  and  $\beta$ ) such that

$$\|M_t(\mu - \mu_\infty)\|_{\phi, \beta} \leq Ce^{-\sigma t} \|\mu - \mu_\infty\|_{\phi, \beta}, \quad \text{for all } t \geq 0,$$

*Remark 2.5.* The constants in Theorem 2.4 can be calculated explicitly. If we set  $\lambda_0 \in [\lambda + 2K/R, 1)$  for any  $\alpha_0 \in (0, \alpha)$  we can choose  $\beta = \alpha_0/K$  and  $\bar{\alpha} = \max\{1 - \alpha - \alpha_0, (2 + R\beta\lambda_0)/(2 + R\beta)\}$ . Then we have  $C := 1/\bar{\alpha}$  and  $\sigma = -\log \bar{\alpha}/T$ .

For the proofs of Theorem 2.3 and Theorem 2.4 we refer to [28, 29] and references therein.

In the following two sections we show how the Foster-Lyapunov condition and the minorisation condition are verified for Equation (5). At the end of the section we give the proof of Theorem 1.3.

We use the notations  $z := (x, v)$  and  $\int dz := \int_{\mathbb{R}^d} \int_{\mathcal{V}} dx dv$  for the rest of the paper whenever convenient.

## 2.1 Foster-Lyapunov condition

In this section, we verify the Foster-Lyapunov condition (24) for Equation (5). In order to look at Lyapunov functions let us fix some notation. We remark that by *Lyapunov functions* we do not refer to scalar functions which are used for stability results in ODE theory. By a Lyapunov function in the sense of Harris's theorem, we want some function  $\phi(z)$  where  $\phi(z) \rightarrow \infty$  as  $|z| \rightarrow \infty$  and the existence of some  $t > 0$ ,  $C > 0$  and  $\alpha \in (0, 1)$  such that

$$\int \phi(z)f(t, z)dz \leq \alpha \int \phi(z)f_0(z) dz + C \int f_0(z) dz, \quad (26)$$

for any initial data  $f_0(z) \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$ .

For  $f$  satisfying an equation

$$\partial_t f = \mathcal{L}f,$$

we can prove (26) by showing that

$$\mathcal{L}^* \phi \leq -\gamma\phi + D, \quad (27)$$

for some positive constants  $\gamma, D$ .

*Remark 2.6.* We take the time derivative of (26) to obtain (27), so that  $\alpha = e^{-\gamma t}$  and  $C = D/\gamma$ .

In (27),  $\mathcal{L}^*$  is the formal adjoint of  $\mathcal{L}$ . In our case

$$\mathcal{L}f = -v \cdot \nabla_x f + \int_{\mathcal{V}} \lambda(v' \cdot \nabla_x M) f(x, v') dv' - \lambda(v \cdot \nabla_x M) f(x, v). \quad (28)$$

Therefore,

$$\mathcal{L}^* \phi = v \cdot \nabla_x \phi + \lambda(v \cdot \nabla_x M) \left( \int_{\mathcal{V}} \phi(x, v') dv' - \phi(x, v) \right). \quad (29)$$

Next, we show the following lemma.

**Lemma 2.7** (Foster-Lyapunov condition for Equation (5)). *Suppose that Hypotheses 1-5 hold. Then there exist constants  $\gamma > 0$  and  $\beta > 0$  such that*

$$\phi(x, v) = (1 - \gamma v \cdot \nabla_x M(x) - \beta \gamma \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x)) e^{-\gamma M(x)},$$

is a function for which the semigroup generated by  $\mathcal{L}$  in (28) satisfies the Foster-Lyapunov condition (24) with  $\beta = \chi/(1 + \chi)$  and

$$\gamma \leq \min \left\{ \frac{\tilde{\lambda} \chi (1 - \chi) \xi}{8(1 + \chi)}, \frac{1 + \chi}{2(2 + \chi) V_0 \|\nabla_x M\|_\infty} \right\},$$

with

$$\xi := \begin{cases} m_*^{k-2}, & \text{if } k < 2, \\ 1, & \text{if } k = 2, \\ \|\nabla_x M\|_\infty^{k-2}, & \text{if } k > 2, \end{cases}$$

where  $m_* > 0$  is found in Hypothesis 3.

*Proof.* We begin by briefly motivation of the form of  $\phi$  in the proof. It is structurally similar to an estimate in Lemma 2.2 in [40]. As the confining terms are bounded, we expect that we need to look for a Foster-Lyapunov functional which has exponential tails, by analogy with parabolic reaction diffusion equations with bounded drift terms. We can also guess this form by looking at the previous results on similar equations including [40]. We choose a function of  $M$  which will have this behaviour,  $e^{-\gamma M}$ , and seek a Foster-Lyapunov functional which is closely related to this. We derive the precise form of  $\phi$  by repeatedly differentiating  $\int f(t, z) e^{-\gamma M(x)} dz$  along the flow of the equation until we find a term which doesn't change sign. We then create our Foster-Lyapunov function from a combination of  $e^{-\gamma M(x)}$  and the key terms appearing in the derivatives of this moment along the flow of the equation.

First we compute the action of  $\mathcal{L}^*$  on the different elements.

$$\mathcal{L}^* \left( e^{-\gamma M(x)} \right) = -\gamma v \cdot \nabla_x M(x) e^{-\gamma M(x)}.$$

Furthermore,

$$\begin{aligned} \mathcal{L}^* \left( v \cdot \nabla_x M(x) e^{-\gamma M(x)} \right) &= (v^T \text{Hess}(M)(x) v - \gamma (v \cdot \nabla_x M(x))^2) e^{-\gamma M(x)} \\ &\quad - ((1 - \chi \psi(v \cdot \nabla_x M(x))) v \cdot \nabla_x M(x)) e^{-\gamma M(x)}. \end{aligned}$$

Lastly,

$$\begin{aligned} \mathcal{L}^* \left( \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x) e^{-\gamma M(x)} \right) &= (\psi'(v \cdot \nabla_x M(x)) v^T \text{Hess}(M)(x) v v \cdot \nabla_x M(x)) e^{-\gamma M(x)} \\ &\quad + (\psi(v \cdot \nabla_x M(x)) v^T \text{Hess}(M)(x) v - \gamma \psi(v \cdot \nabla_x M(x)) (v \cdot \nabla_x M(x))^2) e^{-\gamma M(x)} \\ &\quad + (1 - \chi \psi(v \cdot \nabla_x M(x))) \left( \int_{\mathcal{V}} \psi(v' \cdot \nabla_x M(x)) v' \cdot \nabla_x M(x) dv' - \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x) \right) e^{-\gamma M(x)}. \end{aligned}$$

Putting everything together gives,

$$\begin{aligned} &\mathcal{L}^* \left( (1 - \gamma v \cdot \nabla_x M(x) - \beta \gamma \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x)) e^{-\gamma M(x)} \right) \\ &\leq - \left( \beta \gamma (1 - \chi) \int_{\mathcal{V}} \psi(v' \cdot \nabla_x M(x)) v' \cdot \nabla_x M(x) dv' \right) e^{-\gamma M(x)} \\ &\quad + (\beta \gamma (1 + \chi) - \gamma \chi) \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x) e^{-\gamma M(x)} \\ &\quad + (\gamma^2 (v \cdot \nabla_x M(x))^2 + \gamma^2 \beta \psi(v \cdot \nabla_x M(x)) (v \cdot \nabla_x M(x))^2) e^{-\gamma M(x)} \\ &\quad - (\gamma + \beta \gamma \psi'(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x) + \beta \gamma \psi(v \cdot \nabla_x M(x)) v^T \text{Hess}(M)(x) v) e^{-\gamma M(x)}. \end{aligned}$$

We also have (for  $\beta \leq 1$ )

$$\begin{aligned} & -\gamma - \beta\gamma\psi'(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x)) \\ & \leq \gamma + \beta\gamma \sup_{|z| \leq V_0 \|\nabla_x M\|_\infty} (\psi'(z)z + \psi(z)) \leq \gamma C_1(\psi, \|\nabla_x M\|_\infty), \end{aligned}$$

and

$$\gamma^2(v \cdot \nabla_x M(x))^2 + \gamma^2\beta\psi(v \cdot \nabla_x M(x))(v \cdot \nabla_x M(x))^2 \leq 2\gamma^2|\nabla_x M|^2.$$

Combining these and choosing  $\beta = \chi/(1 + \chi)$  we have,

$$\begin{aligned} & \mathcal{L}^* \left( (1 - \gamma v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x))e^{-\gamma M(x)} \right) \\ & \leq \left( -\beta\gamma\tilde{\lambda}(1 - \chi)|\nabla_x M|^k + 2\gamma^2|\nabla_x M|^2 + \gamma C_1 v^T \text{Hess}(M)(x)v \right) e^{-\gamma M(x)}. \end{aligned}$$

Let us write

$$\xi := \begin{cases} m_*^{k-2}, & \text{if } k < 2, \\ 1, & \text{if } k = 2, \\ \|\nabla_x M\|_\infty^{k-2}, & \text{if } k > 2, \end{cases}$$

where  $m_*$  is coming from Hypothesis 3. Then, if we choose

$$\gamma \leq \min \left\{ \frac{\tilde{\lambda}\chi(1 - \chi)\xi}{8(1 + \chi)}, \frac{1 + \chi}{2(2 + \chi)V_0\|\nabla_x M\|_\infty} \right\},$$

then we have, at least for  $x$  sufficiently large in the case  $k < 2$  that,

$$\begin{aligned} & \mathcal{L}^* \left( (1 - \gamma v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x))e^{-\gamma M(x)} \right) \\ & \leq \gamma \left( -\frac{3\tilde{\lambda}\chi(1 - \chi)}{8(1 + \chi)}|\nabla_x M|^k + C_1 V_0^2 |\text{Hess}(M)(x)| \right) e^{-\gamma M(x)}. \end{aligned}$$

Then by Hypothesis 3 there exist  $R > 0$  and  $m_* > 0$  such that when  $|x| > R$  we have

$$|\nabla_x M| > m_*, \quad \text{and} \quad |\text{Hess}(M)(x)| \leq \frac{\tilde{\lambda}\chi(1 - \chi)m_*^k}{4C_1(1 + \chi)V_0^2}. \quad (30)$$

So we have,

$$\begin{aligned} & \mathcal{L}^* \left( (1 - \gamma v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x))e^{-\gamma M(x)} \right) \\ & \leq A \mathbf{1}_{\{|x| < R\}} - \frac{\gamma\tilde{\lambda}\chi(1 - \chi)m_*^k}{4(1 + \chi)}e^{-\gamma M(x)}, \end{aligned}$$

where

$$A = \sup_{|x| \leq R} \left\{ \gamma C_1 V_0^2 |\text{Hess}(M)(x)| e^{-\gamma M(x)} \right\}. \quad (31)$$

Since we can compare  $e^{-\gamma M(x)}$  to  $(1 - \gamma v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x))e^{-\gamma M(x)}$  by

$$\begin{aligned} (1 - \gamma v \cdot \nabla_x M(x) - \beta\gamma\psi(v \cdot \nabla_x M(x))v \cdot \nabla_x M(x))e^{-\gamma M(x)} & \leq (1 + \gamma V_0 \|\nabla_x M\|_\infty (1 + \beta \|\psi\|_\infty)) e^{-\gamma M(x)} \\ & \leq \frac{3}{2} e^{-\gamma M(x)}, \end{aligned}$$

if we write

$$\phi(x, v) = (1 - \gamma v \cdot \nabla_x M(x) - \beta \gamma \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x)) e^{-\gamma M(x)},$$

then

$$\mathcal{L}^* \phi \leq A - \frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} \phi = -\frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} (A' - \phi), \quad (32)$$

where

$$A' = \frac{6C_1 V_0^2 (1 + \chi)}{\tilde{\lambda} \chi (1 - \chi) m_*^k} \sup_{|x| \leq R} \left\{ |\text{Hess}(M)(x)| e^{-\gamma M(x)} \right\}.$$

Therefore

$$\int f(t, z) \phi(z) dz \leq A' + \exp\left(-\frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} t\right) \left( \int f_0(z) \phi(z) dz - A' \right).$$

Thus we prove (26) for  $\alpha = \exp\left(-\frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} t\right)$  and  $C = A'$ .  $\square$

## 2.2 Minorisation condition

In this section, we show the minorisation condition (25) for Equation (5). We consider two semigroups  $(\mathcal{T}_t)_{t \geq 0}$  and  $(\mathcal{S}_t)_{t \geq 0}$ . Let  $(\mathcal{T}_t)_{t \geq 0}$ , represents the *transport* part, be associated to the equation

$$\partial_t f + v \cdot \nabla_x f + \lambda(x, v) f = 0, \quad (33)$$

which means that the solution of (33) can be written as for  $t \geq 0$

$$\mathcal{T}_t f_0(x, v) = \begin{cases} f_0(x - vt, v) \exp\left(-\int_0^t \lambda(x - vs, v) ds\right), & x \geq vt \\ 0, & x < vt. \end{cases} \quad (34)$$

Let  $(\mathcal{S}_t)_{t \geq 0}$  be associated to the equation

$$\partial_t f + v \cdot \nabla_x f + \lambda(v, x) f = \int_{\mathcal{V}} \lambda(x, v') f(t, x, v') dv'. \quad (35)$$

Then the solution of (35) is

$$f(t, x, v) = \mathcal{S}_t f_0(x, v) = \mathcal{T}_t f_0(x, v) + \int_0^t \mathcal{T}_{t-s} (\mathcal{J} f(s, x, v)) ds,$$

where  $\mathcal{J} f(t, x, v) := \int_{\mathcal{V}} \lambda(x, v') f(t, x, v') dv'$  is the *jump operator*. Remark that we have

$$\mathcal{J} f(t, x, v) = \int_{\mathcal{V}} \lambda(x, v') f(t, x, v') dv' \geq (1 - \chi) \mathbb{1}_{\{|v| \leq V_0\}} \int_{\mathcal{V}} f(t, x, v') dv'. \quad (36)$$

By applying Duhamel's formula iteratively we obtain

$$f(t, x, v) = \mathcal{S}_t f_0(x, v) \geq (1 - \chi)^2 e^{-(1+\chi)t} \int_0^t \int_0^s \mathcal{T}_{t-s} \mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0(x, v) dr ds.$$

**Lemma 2.8.** *Given any time  $t_0 > 0$ , for all  $t \geq t_0$  it holds that*

$$\int_{\mathcal{V}} \mathcal{T}_t (\delta_{x_0}(x) \mathbb{1}_{\{|v_0| \leq V_0\}}(v)) dv \geq e^{-(1+\chi)t} \frac{1}{t^d |B(V_0)|} \mathbb{1}_{\{|x-x_0| \leq V_0 t\}} \quad \text{for any } x_0, v_0 > 0.$$

*Proof.* Note that we have

$$\mathcal{T}_t f_0(x, v) \geq e^{-(1+\chi)t} f_0(x - vt, v), \quad t \geq 0.$$

For an arbitrary starting point and a velocity  $(x_0, v_0)$ ,  $x_0 > 0$ ,  $v_0 \in B(V_0)$  (ball of radius  $V_0$ ) we have

$$\mathcal{T}_t (\delta_{x_0}(x) \mathbb{1}_{\{|v_0| \leq V_0\}}(v)) \geq e^{-(1+\chi)t} \delta_{x_0}(x - vt) \mathbb{1}_{\{|v_0| \leq V_0\}}.$$

By integrating this and changing variables we obtain

$$\begin{aligned} \int_{\mathcal{V}} \mathcal{T}_t (\delta_{x_0}(x) \mathbb{1}_{\{|v_0| \leq V_0\}}) \, dv &\geq e^{-(1+\chi)t} \int_{\mathcal{V}} \delta_{x_0}(x - vt) \mathbb{1}_{\{|v_0| \leq V_0\}}(v) \, dv \\ &\geq e^{-(1+\chi)t} \frac{1}{t^d |B(V_0)|} \int_{|\frac{x-y}{t}| \leq V_0} \delta_{x_0}(y) \mathbb{1}_{\{|\frac{x-y}{t}| \leq V_0\}}(v) \, dy. \end{aligned}$$

This gives the result.  $\square$

Now, we prove the minorisation condition for (5) below.

**Lemma 2.9** (Minorisation condition for Equation (5)). *For every  $R_* > 0$  we can take  $t = 3 + R_*/V_0$  such that any solution of Equation (5) with initial data  $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathcal{V})$  with  $\int_{|x| \leq R_*} \int_{\mathcal{V}} f_0(x, v) \, dx \, dv = 1$  satisfies*

$$f(t, x, v) \geq (1 - \chi^2) e^{-(1+\chi)t} \frac{1}{t^d |B(V_0)|} \mathbb{1}_{\{|x| \leq V_0\}} \mathbb{1}_{\{|v| \leq V_0\}}. \quad (37)$$

*Proof.* We take  $f_0(x, v) := \delta_{(x_0, v_0)}$  where  $(x_0, v_0) \in \mathbb{R}^d \times \mathcal{V}$ , is an arbitrary point with an arbitrary velocity. We only need to consider  $x_0 \in B(0, R_*)$ , then the bound we obtain depends on  $R_*$ . First we have that

$$\mathcal{T}_r f_0 \geq e^{-(1+\chi)r} \delta_{(x_0 + rv_0, v_0)}.$$

Applying  $\mathcal{J}$  to this we get

$$\mathcal{J} \mathcal{T}_r f_0 \geq (1 - \chi) e^{-(1+\chi)r} \delta_{x_0 + rv_0}(x) \mathbb{1}_{\{|v| \leq V_0\}}.$$

We then apply Lemma 2.8 and obtain

$$\int_{\mathcal{V}} T_{s-r} \mathcal{J} \mathcal{T}_r f_0 \geq (1 - \chi) e^{-(1+\chi)s} \frac{1}{(s-r)^d |B(V_0)|} \mathbb{1}_{\{|x - x_0 - rv_0| \leq V_0(s-r)\}}.$$

This means that

$$\mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0 \geq (1 - \chi)^2 e^{-(1+\chi)s} \frac{1}{(s-r)^d |B(V_0)|} \mathbb{1}_{\{|x - x_0 - rv_0| \leq V_0(s-r)\}} \mathbb{1}_{\{|v| \leq V_0\}}.$$

Lastly we have that

$$\mathcal{T}_{t-s} \mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0 \geq (1 - \chi)^2 e^{-(1+\chi)t} \frac{1}{(s-r)^d |B(V_0)|} \mathbb{1}_{\{|x - (t-s)v - x_0 - rv_0| \leq V_0(s-r)\}} \mathbb{1}_{\{|v| \leq V_0\}}.$$

Since we have (remembering that all the velocities are smaller than  $V_0$ )

$$|x - v(t-s) - x_0 - rv_0| \leq (s-r)V_0,$$

implies that

$$|x| \leq (s-r)V_0 - (t-s)V_0 - rV_0 - R_*.$$

Then if we ensure that  $(s-r) \geq 2 + R_*/V_0$ ,  $r \leq 1/2$  and  $(t-s) \leq 1/2$  we will have

$$\mathcal{T}_{t-s} \mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0 \geq (1-\chi)^2 e^{-(1+\chi)t} \frac{1}{(s-r)^d |B(V_0)|} \mathbb{1}_{\{|x| \leq V_0\}} \mathbb{1}_{\{|v| \leq V_0\}}.$$

Therefore let us set  $t = 3 + R_*/V_0$ . Then we can restrict the time integrals to  $r \in (0, 1/2)$ ,  $s \in (5/2 + R_*/V_0, 3 + R_*/V_0)$ . Then we get

$$\begin{aligned} f(t, x, v) &\geq \int_0^t \int_0^s \mathcal{T}_{t-s} \mathcal{J} \mathcal{T}_{s-r} \mathcal{J} \mathcal{T}_r f_0(x, v) \, dr \, ds \\ &\geq (1-\chi)^2 e^{-(1+\chi)t} \int_{5/2+R_*/V_0}^{3+R_*/V_0} \int_0^{1/2} \frac{1}{(s-r)^d |B(V_0)|} \mathbb{1}_{\{|x| \leq V_0\}} \mathbb{1}_{\{|v| \leq V_0\}} \, dr \, ds \\ &\geq (1-\chi)^2 e^{-(1+\chi)t} \frac{1}{t^d |B(V_0)|} \mathbb{1}_{\{|x| \leq V_0\}} \mathbb{1}_{\{|v| \leq V_0\}}. \end{aligned}$$

This gives the uniform lower bound we need for Harris's theorem. We can extend this from delta function initial data to general initial data by using the fact that the associated semigroup is Markov.  $\square$

*Proof of Theorem 1.3.* We verify the two hypotheses of Harris's theorem in Lemmas 2.7 and 2.9. The contraction in the  $\|\cdot\|_*$  norm and the existence of a steady state follow again by Harris's theorem.

Moreover Lemma 2.7 gives that for the steady state  $f_\infty$  obtained by Harris's theorem we have

$$\int \phi(z) f_\infty(z) \, dz \leq A'.$$

Our conditions on  $\gamma$  ensure that

$$\frac{1}{2} e^{-\gamma M(x)} \leq \phi \leq \frac{3}{2} \phi.$$

Therefore we obtain

$$\int e^{-\gamma M(x)} f_\infty(z) \, dz \leq 2A',$$

and this leads to

$$\int e^{-\gamma M(x)} f(t, z) \, dz \leq 2A' + 3 \exp\left(-\frac{\gamma \tilde{\lambda} \chi (1-\chi) m_*^k}{6(1+\chi)} t\right) \int e^{-\gamma M(x)} f_0(z) \, dz,$$

which gives the contraction in the  $\|\cdot\|_{**}$  norm. We remark that in this proof  $\gamma$  only depends on  $M$  through  $\tilde{\lambda}$  and  $\|\nabla_x M\|_\infty$ . So if  $\psi'(0) > 0$  we can choose  $\gamma$  uniformly over sets of  $M$  where  $\nabla_x M$  is bounded uniformly.  $\square$

### 3 Weakly non-linear coupling

#### 3.1 Stationary solutions

In this section, we build a stationary state for the run and tumble equation (5) with the weakly non-linear coupling (10). We know by Theorem 1.3 that there exists a unique steady state solution to the linear equation satisfying the assumptions listed in Theorem 1.3. For each fixed  $M$ , we call  $\mathcal{S}_t^M$  the semigroup on measures associated to the linear equation and  $f_\infty^M$  its unique stationary solution. Then we see that  $f_\infty^M$  satisfies

$$v \cdot \nabla_x f_\infty^M(x, v) + \lambda(v \cdot \nabla_x M(x)) f_\infty^M(x, v) - \int \lambda(v' \cdot \nabla_x M(x)) f_\infty^M(x, v') dv' = 0. \quad (38)$$

We define a function  $G : C^2(\mathbb{R}) \rightarrow C^2(\mathbb{R})$  given by

$$G(M) = \log(S_\infty(1 + \eta N * \rho^M)), \quad (39)$$

where  $S_\infty$  a smooth function, having exponential tails with some fixed parameter,  $\eta > 0$  a small constant,  $N$  a positive, compactly supported, smooth function, and  $\rho^M := \int f_\infty^M(x, v) dv'$ . We see that if  $M$  is a fixed point of  $G$  then  $f_\infty^M$  will be a steady state of the non-linear equation.

**Proposition 3.1.** *Suppose that  $M$  is of the form  $M = M_\infty + \log(1 + \eta N * \rho)$  for some  $\rho \in \mathcal{P}(\mathbb{R}^d)$ . Then if  $\eta$  is small enough in terms of  $\|N\|_{W^{2,\infty}}$ , we have that*

$$\|\mathcal{S}_t^M f\|_{**} \leq D e^{-\sigma t} \|f\|_{**},$$

where  $D, \sigma$  are strictly positive constants only depending on  $M_\infty, N$ , and  $\eta$ . Furthermore, if  $f_\infty^M$  is the steady state of  $\mathcal{S}_t^M$  then

$$\|f_\infty^M\|_{**} \leq \tilde{C}, \quad (40)$$

where  $\tilde{C}$  is a constant depending on  $M_\infty, N$  and  $\eta$ .

*Proof.* The result follows from Theorem 1.3. We recall that the constants in Lemma 2.9 in the minorisation part do not depend on  $M$ , whereas, the constants in Lemma 2.7 in the Foster-Lyapunov part depend on  $M$  through  $\|\nabla_x M\|_\infty, R$ , and  $m_*$  so that for all  $|x| > R$  we have (recalling (30)),

$$|\nabla_x M| > m_*, \quad \text{and} \quad |\text{Hess}(M)| \leq \frac{\tilde{\lambda} \chi (1 - \chi) m_*^k}{4C_1(1 + \chi)V_0^2}.$$

We want to verify this for  $M$  solving (10). We can control  $|\nabla_x M|$  and  $|\text{Hess}(M)|$  by considering

$$M = M_\infty + \log(1 + \eta N * \rho) \sim M_\infty + \eta N * \rho^M.$$

Provided that  $\eta \leq \|N\|_\infty^{-1}$ , which we can choose it to be, by Taylor expansion we have that

$$|M - M_\infty| \leq \eta N * \rho \leq \eta \|N\|_\infty.$$

In a similar way, we can take gradients to get

$$\nabla_x M = \nabla_x M_\infty + \eta \frac{\nabla_x N * \rho}{1 + \eta N * \rho}.$$

Then

$$\left\| \frac{\nabla_x N * \rho}{1 + \eta N * \rho} \right\|_\infty \leq \|\nabla_x N * \rho\|_\infty \leq \|\nabla_x N\|_\infty.$$

So we can ensure that

$$|\nabla_x M - \nabla_x M_\infty| \leq \eta \|\nabla_x N\|_\infty. \quad (41)$$

We can also compute the Hessian to get

$$\text{Hess}(M) = \text{Hess}(M_\infty) + \frac{\eta(\text{Hess}(N) * \rho) + \eta^2 ((N * \rho)(\text{Hess}(N) * \rho) - (\nabla_x N * \rho)(\nabla_x N^T * \rho))}{(1 + \eta N * \rho)^2}.$$

Therefore, the difference between  $\text{Hess}(M)$  and  $\text{Hess}(M_\infty)$  is controlled by  $\eta \|N\|_{W^{2,\infty}}$ . Suppose that there exist  $R_\infty$  and  $m_\infty$  such that for all  $|x| > R_\infty$  we have

$$|\nabla_x M_\infty| \geq m_\infty, \quad \text{and} \quad |\text{Hess}(M_\infty)| \leq \frac{\tilde{\lambda}\chi(1-\chi)m_\infty^k}{32C_1(1+\chi)V_0^2}.$$

Then by choosing  $\eta$  small enough in terms of  $\|N\|_{W^{2,\infty}}$  and setting  $m_* = m_\infty/2$  and  $R = R_\infty$  we have  $m_*$  and  $R$  in (30) only depend on  $M_\infty, N, \eta$ .

Furthermore, by Theorem 1.3 for the steady state  $f_\infty^M$  we have

$$\int e^{-\gamma M(x)} f_\infty^M(z) dz \leq 2A',$$

where

$$A' = \frac{6C_1V_0^2(1+\chi)}{\tilde{\lambda}\chi(1-\chi)m_*^k} \sup_{|x| \leq R} \left\{ |\text{Hess}(M)(x)| e^{-\gamma M(x)} \right\}.$$

We can bound  $A'$  only in terms of  $M_\infty, N, \eta$ . We already know this is true for  $m_*$  and  $R$ . Moreover, as  $\gamma \leq 1$  we have

$$\sup_{|x| \leq R} \left\{ |\text{Hess}(M)(x)| e^{-\gamma M(x)} \right\} \leq \sup_{|x| \leq R} \left\{ (|\text{Hess}(M_\infty)(x)| + \eta \|N\|_{W^{2,\infty}}) e^{-M_\infty(x) + \eta \|N\|_\infty} \right\},$$

which we can bound in a way that only depends on  $M_\infty, N, \eta$ . Therefore,

$$\int e^{-\gamma M_\infty(x)} f_\infty^M(z) dz \leq e^{\eta \|N\|_\infty} \int e^{-\gamma M(x)} f_\infty^M(z) dz,$$

and we can compare  $\gamma M_\infty(x)$  to  $\delta$  in Theorem 1.3. So this lets us control  $\|f_\infty^M\|_{**}$  in terms of  $A'$  up to factors only depending on  $M_\infty, N, \eta$ . This finishes the proof.  $\square$

Then we can prove

**Proposition 3.2.** *We consider Equation (5) with the weakly non-linear coupling (10) where we suppose that  $N$  is a positive, smooth function with a compact support,  $\eta > 0$  is a constant, and  $S_\infty$  is a smooth function satisfying for some  $C_1, C_2, \alpha > 0$  that*

$$C_1 - \alpha \langle x \rangle \leq M_\infty(x) := \log(S_\infty(x)) \leq C_2 - \alpha \langle x \rangle, \quad (42)$$

where  $\langle x \rangle = \sqrt{1+x^2}$ . Then there exists some constant  $\tilde{C}$  depending on  $C_1, C_2, \alpha$  such that if  $\eta < \tilde{C}$  then there exists a unique steady state solution to Equation (5) with a weakly non-linear coupling.

*Proof.* We want to use the contraction mapping theorem to show that  $G$ , defined by (39), has a fixed point. Let us take for  $i = \{1, 2\}$ ,

$$M_i = M_\infty + \log(1 + \eta N * \rho^{M_i}), \quad \text{where} \quad \rho^{M_i} = \int_{\mathcal{V}} f_\infty^{M_i}(x, v) dv.$$

We also know that  $M_\infty$  satisfies (42). Then we show contractivity of  $G$  by using the fact that

$$\|G(M_1) - G(M_2)\|_\infty \leq C\eta \|N * \rho^{M_1} - N * \rho^{M_2}\|_\infty \leq C\eta \|N\|_\infty \|\rho^{M_1} - \rho^{M_2}\|_{**},$$

where  $C > 0$  is a constant.

Let us call  $\mathcal{S}_t^{M_i}$ , for  $i = \{1, 2\}$ , the semigroups associated to the linear equation with  $M_i := \log S_i$ . Then, we choose  $t$  sufficiently large so that  $\mathcal{S}_t^{M_1}$  is a contraction. By Proposition 3.1 we know that there exist  $D, \sigma > 0$  such that

$$\|\mathcal{S}_t^{M_1}(f - g)\|_{**} \leq D e^{-\sigma t} \|f - g\|_{**}.$$

The constants  $D, \sigma$  only depend on  $M_\infty, N, \eta$  because it was shown in Lemma 3.1, the bounds on  $M$  required to prove Theorem 1.3 are preserved by  $G$  and do not depend on  $M$  except through,  $M_\infty, N, \eta$ . We recall

$$\|f\|_{**} = \int_{\mathbb{R}^d} \int_{\mathcal{V}} e^{\delta \langle x \rangle} |f(t, x, v)| dx dv,$$

where  $\delta = \beta\gamma$ . Note that the definition of  $\delta$  comes from the fact that we essentially weight by  $e^{-\gamma M_\infty(x)}$  and  $M_\infty(x) \sim -\beta \langle x \rangle$ . Let us call  $f_\infty^{M_i}$  are the steady state solutions of the linear equation with  $M_i$  for  $i = \{1, 2\}$ . Then

$$\|f_\infty^{M_1} - f_\infty^{M_2}\|_{**} = \|\mathcal{S}_t^{M_1} f_\infty^{M_1} - \mathcal{S}_t^{M_2} f_\infty^{M_2}\|_{**} \leq \|\mathcal{S}_t^{M_1}(f_\infty^{M_1} - f_\infty^{M_2})\|_{**} + \|(\mathcal{S}_t^{M_1} - \mathcal{S}_t^{M_2})f_\infty^{M_2}\|_{**}$$

leading to

$$(1 - D e^{-\sigma t}) \|f_\infty^{M_1} - f_\infty^{M_2}\|_{**} \leq \|(\mathcal{S}_t^{M_1} - \mathcal{S}_t^{M_2})f_\infty^{M_2}\|_{**}. \quad (43)$$

So it only remains to show that for a fixed time period,  $\mathcal{S}_t^M$  is continuous in  $M$ .

Let us write

$$\Lambda(s, t, M_i)(x, v) = \int_s^t \lambda(v \cdot \nabla_x M_i(x - v(t - r))) dr,$$

and

$$\mathcal{J}^{M_i}(f)(x, v) = \int_{\mathcal{V}} \lambda(v' \cdot \nabla_x M_i(x)) f(x, v') dv'.$$

Then we have

$$\mathcal{S}_t^{M_i} f = e^{-\Lambda(0, t, M_i)} \mathcal{T}_t f + \int_0^t e^{-\Lambda(s, t, M_i)} \mathcal{J}^{M_i} \mathcal{T}_{t-s} \mathcal{S}_s^{M_i} f ds,$$

where  $(\mathcal{T})_{t \geq 0}$  is defined in (35). Consequently we have

$$\begin{aligned} |\mathcal{S}_t^{M_1} f - \mathcal{S}_t^{M_2} f| &\leq \left( e^{-\Lambda(0, t, M_1)} - e^{-\Lambda(0, t, M_2)} \right) \mathcal{T}_t f \\ &\quad + \int_0^t \left( e^{-\Lambda(s, t, M_1)} - e^{-\Lambda(s, t, M_2)} \right) \mathcal{J}^{M_1} \mathcal{T}_{t-s} \mathcal{S}_s^{M_1} f ds \\ &\quad + \int_0^t e^{-\Lambda(s, t, M_2)} (\mathcal{J}^{M_1} - \mathcal{J}^{M_2}) \mathcal{T}_{t-s} \mathcal{S}_s^{M_1} f ds \\ &\quad + \int_0^t e^{-\Lambda(s, t, M_2)} \mathcal{J}^{M_2} (\mathcal{S}_s^{M_1} - \mathcal{S}_s^{M_2}) f ds. \end{aligned}$$

We can see that for  $s, t \leq T$  there exists a constant  $C_T > 0$  depending on  $T$  so that

$$\left| e^{-\Lambda(s,t,M_1)} - e^{-\Lambda(s,t,M_2)} \right| \leq C_T \|\nabla_x M_1 - \nabla_x M_2\|_\infty.$$

We also have trivially that

$$e^{-\Lambda(s,t,M)} \leq 1.$$

Turning to the jump operator  $\mathcal{J}$  we have

$$\|(\mathcal{J}^{M_1} - \mathcal{J}^{M_2})f\|_{**} \leq \|\lambda(v \cdot \nabla_x M_1) - \lambda(v \cdot \nabla_x M_2)\|_\infty \|f\|_{**} \leq C \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|f\|_{**},$$

and

$$\|\mathcal{J}^{M_i} f\|_{**} \leq (1 + \chi) \|f\|_{**}.$$

We also have

$$\|\mathcal{T}_t f\|_{**} \leq e^{2\delta V_0 t} \|f\|_{**}.$$

Therefore we obtain, for  $t \leq T$ ,

$$\left\| \left( \mathcal{S}_t^{M_1} - \mathcal{S}_t^{M_2} \right) f \right\|_{**} \leq C_T \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|f\|_{**} + \int_0^t C_T \left\| \left( \mathcal{S}_s^{M_1} - \mathcal{S}_s^{M_2} \right) f \right\|_{**} ds$$

Then Gronwall's inequality gives,

$$\left\| \left( \mathcal{S}_t^{M_1} - \mathcal{S}_t^{M_2} \right) f \right\|_{**} \leq C'_T \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|f\|_{**}, \quad (44)$$

where  $C'_T > 0$  a constant depending on  $T$ .

Using (43) and (44) we obtain an estimate on the steady states given by

$$\|f_\infty^{M_1} - f_\infty^{M_2}\|_{**} \leq (1 - De^{-\sigma T})^{-1} C'_T \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|f_\infty^{M_2}\|_{**}.$$

Now we can see that

$$\|\rho^{M_1} - \rho^{M_2}\|_{**} = \|f_\infty^{M_1} - f_\infty^{M_2}\|_{**}. \quad (45)$$

Consequently we have,

$$\|G(M_1) - G(M_2)\|_\infty \leq C\eta \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|\rho^{M_2}\|_{**}.$$

Similarly

$$\|\nabla_x G(M_1) - \nabla_x G(M_2)\|_\infty \leq C\eta \|\nabla_x M_1 - \nabla_x M_2\|_\infty \|\rho^{M_2}\|_{**}.$$

By Proposition 3.1, we also have that

$$\|\rho^{M_2}\|_{**} = \|f_\infty^{M_2}\|_{**} \leq \tilde{C}.$$

So we choose  $\eta$  sufficiently small to get

$$\|G(M_1) - G(M_2)\|_{W^{1,\infty}} \leq \frac{1}{2} \|M_1 - M_2\|_{W^{1,\infty}}.$$

This gives a unique fixed point of  $G$ ; and, as for the weakly non-linear equation, a steady state solution.  $\square$

### 3.2 Perturbation argument

In this section, we prove that the solution of Equation (5) with the weakly non-linear coupling (10) converges exponentially to its unique steady state solution obtained in Proposition (3.2).

Let us call  $\tilde{M}$  the fixed point of  $G$  we found in Proposition 3.2 and  $f_\infty = f^{\tilde{M}}$  the steady state of the linear equation associated to  $\tilde{M}$  which is by construction also the steady state of the weakly non-linear equation. We showed, in Proposition 3.1, that we can find  $R$ ,  $m_*$  and bound  $\|\nabla_x M\|_\infty$  uniformly over the set of log-chemoattractants of the form

$$M = M_\infty + \log(1 + \eta N * \rho),$$

for some probability density  $\rho$  on  $\mathbb{R}^d$ . This means that we can also fix,  $\gamma$  and  $\tilde{\lambda}$  uniformly over this set since we show in the proof of Proposition 3.1 that they only depend on these bounds.

Let us first look at a moment estimate for the weakly non-linear equation (5)-(10). We would like to show an inequality analogous to (26) for the solution  $f$  of the weakly non-linear equation. That is to say we show

$$\int e^{-\gamma M_\infty(x)} f(t, z) dz \leq \alpha \int e^{-\gamma M_\infty(x)} f_0(z) dz + C \int f_0(z) dz. \quad (46)$$

Let us define two operators  $\mathcal{L}_{M_t}$  and  $\mathcal{L}_{M_\infty}$  associated to the weakly non-linear equation and the equation for the stationary solution (38) respectively. Then we have

$$\mathcal{L}_{M_t} f = -v \cdot \nabla_x f + \int \lambda(v' \cdot \nabla_x M) f(t, x, v') dv' - \lambda(v \cdot \nabla_x M) f(t, x, v), \quad (47)$$

where  $M$  is given by (10). Similarly  $\mathcal{L}_{M_\infty}$  is given by

$$\mathcal{L}_{M_\infty} f = -v \cdot \nabla_x f + \int \lambda(v' \cdot \nabla_x M_\infty) f(t, x, v') dv' - \lambda(v \cdot \nabla_x M_\infty) f(t, x, v). \quad (48)$$

We carry out a similar argument to the one in Section 2.1 for the linear equation. We show

**Lemma 3.3.** *Suppose that  $\mathcal{L}_{M_t}$  and  $\mathcal{L}_{M_\infty}$  are given by (47) and (48) and  $\mathcal{L}_{M_t}^*$ ,  $\mathcal{L}_{M_\infty}^*$  denote their formal adjoints respectively. Then let*

$$\phi(x, v) = (1 - \gamma v \cdot \nabla_x M(x) - \beta \gamma \psi(v \cdot \nabla_x M(x)) v \cdot \nabla_x M(x)) e^{-\gamma M_\infty(x)}, \quad (49)$$

and  $M_t = M_\infty + \log(1 + \eta N * \rho_t)$  where  $\rho_t := \int_{\mathcal{V}} f(t, x, v) dv$ . Then we have

$$\mathcal{L}_{M_t}^* \phi \leq \mathcal{L}_{M_\infty}^* \phi + 4\eta \chi V_0 \|\psi'\|_\infty \|\nabla_x N\|_\infty e^{-\gamma M_\infty(x)}. \quad (50)$$

*Proof.* First, using (41) we obtain

$$|\psi(v \cdot \nabla_x M_\infty) - \psi(v \cdot \nabla_x M)| \leq \|\psi'\|_\infty |v| |\nabla_x M - \nabla_x M_\infty| \leq \eta V_0 \|\psi'\|_\infty \|\nabla_x N\|_\infty. \quad (51)$$

Then, we see that

$$\begin{aligned} \mathcal{L}_{M_t}^* \phi - \mathcal{L}_{M_\infty}^* \phi &= (\lambda(v \cdot \nabla_x M_t) - \lambda(v \cdot \nabla_x M_\infty)) \left( \int_{\mathcal{V}} \phi(x, v') dv' - \phi(x, v) \right) \\ &= \chi (\psi(v \cdot \nabla_x M_t) - \psi(v \cdot \nabla_x M_\infty)) \left( \int_{\mathcal{V}} \phi(x, v') dv' - \phi(x, v) \right) \\ &\leq 4\eta \chi V_0 \|\psi'\|_\infty \|\nabla_x N\|_\infty e^{-\gamma M_\infty(x)}. \end{aligned}$$

In the last line of the above inequality, we used the fact that  $\gamma$  is chosen so that  $\phi \leq 2e^{-\gamma M_\infty(x)}$ . This gives (50).  $\square$

**Lemma 3.4.** *Let  $f$  be the solution of Equation (5) with the coupling (10). If  $\eta$  is sufficiently small, then there exists a constant  $B > 0$  (not depending on  $\eta$ ) such that*

$$\int \phi(z)f(t, z) dz \leq \frac{A}{B} + e^{-Bt} \int \phi(z)f_0(z) dz, \quad (52)$$

where  $A$  is given by (31) in the proof of Lemma 2.7 and  $\phi$  is given in (49). In fact we have the bound

$$\|f\|_* \leq \frac{A}{B} + \|f_0\|_*. \quad (53)$$

Using equivalence of norms we also have

$$\|f\|_{**} \leq C^* + 4\|f_0\|_{**},$$

for  $C^* > 0$  a constant.

*Proof.* From Lemma 2.7, inequality (32) we know that

$$\mathcal{L}_{M_\infty}^* \phi \leq A - \frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} e^{-\gamma M_\infty(x)}.$$

Using (50) in Lemma 3.3 we obtain

$$\mathcal{L}_{M_t}^* \phi \leq A - \left( \frac{\gamma \tilde{\lambda} \chi (1 - \chi) m_*^k}{6(1 + \chi)} - 4\eta \chi V_0 \|\psi'\|_\infty \|\nabla_x N\|_\infty \right) e^{-\gamma M_\infty(x)}.$$

Therefore, if we take  $\eta$  such that

$$\eta \leq \frac{\tilde{\lambda} \chi (1 - \chi) m_*^k}{48 \chi (1 + \chi) V_0 \|\psi'\|_\infty \|\nabla_x N\|_\infty},$$

then we have for some constant  $B > 0$

$$\frac{d}{dt} \int \phi(z)f(t, z) dz \leq -B \int \phi(z)f(t, z) dz + A \int f_0(z) dz.$$

Therefore, by Gronwall's inequality we obtain (52). We can also turn this into an exponential decay on

$$\int e^{-\gamma M_\infty(x)} f(t, z) dz.$$

This gives the result. □

**Lemma 3.5.** *Suppose that  $f_t$  is the solution of Equation (5) with the coupling (10) and  $f_\infty$  its steady state solution. Suppose that  $\eta$  is small enough so that Lemmas 3.3 and 3.4 are valid. Suppose also that*

$$\|f_0\|_{**} < \frac{1}{4} \left( \frac{\sigma^2}{4\eta \chi V_0 D \|\psi'\|_\infty \|\nabla_x N\|_\infty} - C^* \right), \quad (54)$$

where  $\sigma, D$  and  $C^*$  are found in Theorem 1.3, Proposition 3.1 and Lemma 3.4 respectively. Then we have for some  $C > 0$  that

$$\|f_t - f_\infty\|_{**} \leq C e^{-\sigma t/2} \|f_0 - f_\infty\|_{**}.$$

*Proof.* We rewrite the weakly non-linear equation (5)-(10) as

$$\partial_t f(t, x, v) = \mathcal{L}_{M_t} f(t, x, v) = \mathcal{L}_{\tilde{M}} f(t, x, v) - (\mathcal{L}_{\tilde{M}} - \mathcal{L}_{M_t}) f(t, x, v),$$

where  $\tilde{M}$  is the fixed point of  $G$  which is defined in (39). Let us call the last term  $h = h(t, x, v) := (\mathcal{L}_{\tilde{M}} - \mathcal{L}_{M_t}) f$ . Then by Duhamel's formula we have

$$f_t = f(t, x, v) = \mathcal{S}_t^{\tilde{M}} f_0(x, v) + \int_0^t \mathcal{S}_{t-s}^{\tilde{M}} h(s, x, v) ds. \quad (55)$$

where  $(\mathcal{S}_t^{\tilde{M}})_{t \geq 0}$  is the semigroup associated to Equation (35). Using definitions (47) and (48) we have

$$h(t, x, v) = \chi \left( \int_{\mathcal{V}} \left( \psi(v' \cdot \nabla_x M_t) - \psi'(v' \cdot \nabla_x \tilde{M}) \right) f(t, x, v') dv' - \left( \psi(v \cdot \nabla_x M_t) - \psi(v \cdot \nabla_x \tilde{M}) \right) f \right).$$

Then, using (41) and (45) from Propositions 3.1 and 3.2 respectively, we have

$$\begin{aligned} \|h\|_{**} &\leq 2\chi \|\psi'\|_{\infty} V_0 \|\nabla_x M_t - \nabla_x \tilde{M}\|_{\infty} \|f\|_{**} \\ &\leq 2\chi \|\psi'\|_{\infty} V_0 \|\nabla_x \log(1 + \eta N * \rho) - \nabla_x \log(1 + \eta N * \rho_{\infty})\|_{\infty} \|f\|_{**} \\ &\leq 2\chi \eta V_0 \|\psi'\|_{\infty} \|\nabla_x N\|_{\infty} \|f_t - f_{\infty}\|_{**} \|f\|_{**}. \end{aligned}$$

Therefore we obtain

$$\|h\|_{**} \leq C\eta \|f_t - f_{\infty}\|_{**} \|f_t\|_{**} \quad (56)$$

where  $C$  is a constant depending on  $\chi, \psi, V_0, N$ . Now we subtract  $f_{\infty}$  from both sides of (55) and take the norms to get

$$\|f_t - f_{\infty}\|_{**} = \|\mathcal{S}_t^{\tilde{M}} f_0 - f_{\infty}\|_{**} + \left\| \int_0^t \mathcal{S}_{t-s}^{\tilde{M}} h(s) ds \right\|_{**}. \quad (57)$$

We can bound the first term in the right hand side of (57) by the result of Theorem 1.3 and the second term by (56). Therefore we obtain

$$\|f_t - f_{\infty}\|_{**} \leq C_1 e^{-\sigma t} \|f_0 - f_{\infty}\|_{**} + C\eta \int_0^t e^{-\sigma(t-s)} \|f_s - f_{\infty}\|_{**} \|f_t\|_{**} ds,$$

where  $C > 0$ , the constant in (56), depends on  $\chi, \psi, V_0, N$ . By the constraint (54) on  $\eta$ , and the bound on  $\|f_t\|_{**}$  from Lemma 3.4 we have

$$\|f_t - f_{\infty}\|_{**} \leq C_1 e^{-\sigma t} \|f_0 - f_{\infty}\|_{**} + \frac{\sigma}{2} \int_0^t e^{-\sigma(t-s)} \|f_s - f_{\infty}\|_{**} ds.$$

By Gronwall's inequality this leads to

$$\|f_t - f_{\infty}\|_{**} \leq C e^{-\sigma t/2} \|f_0 - f_{\infty}\|_{**}$$

for some constant  $C > 0$ . This finishes the proof.  $\square$

*Proof of Theorem 1.4.* Proposition 3.2 gives a unique steady state solution for the weakly non-linear equation (5)-(10). The exponential relaxation to the steady state solution follows from Lemma 3.5. This completes the proof.  $\square$

## 4 Discussion and future research

In this section we discuss the relationship of our work to the much more challenging problem of finding steady states to the run and tumble equation with the fully nonlinear coupling of the form

$$-\Delta S + S = \rho.$$

Note that, this corresponds to the case where the chemical degradation rate  $\alpha = 1$  in (9) for simplicity. Our goal is to describe hopeful direction for future research as well as giving an idea of why we consider the weakly non-linear coupling studied here as a possible stepping stone towards this more complex model. In this regard, we believe that a Schauder fixed point argument is a plausible strategy for finding a steady state of the fully non-linear coupling. We suggest looking for fixed points of the following function  $\tilde{G}(M) = \log S$  where  $S$  is the solution to

$$-\Delta_x S + S = \rho^M,$$

where  $\rho^M$  is the spatial marginal of the unique steady state of (5) with the log-chemoattractant  $M$ .

The first step is to determine if the estimates we obtain in Section 2.1 (the Foster-Lyapunov part) would be good enough to run such a fixed point argument, that is, we would like to see if the bounds we find on

$$\int f^M \phi \, dz,$$

are sufficient to find a compact, convex set of possible chemoattractant densities which is preserved by  $\tilde{G}$ . Since there is a one-to-one correspondence between  $\tilde{G}(M)$  and  $\rho^M$ , this is equivalent to finding a set of possible  $\rho^M$ . A standard way of showing the necessary compactness would be to show tightness of the measures  $\rho^M$ , and this can be achieved by proving moment estimates (such as are found in the Foster-Lyapunov part). However, we encounter a problem that at each iteration of such a scheme we lose weight in our moment estimate. Suppose that we have a spatial density  $\rho$  which we know satisfies a moment bound of the form

$$\int e^{\alpha\langle x \rangle} \rho(x) \, dx \leq C,$$

for some constants  $\alpha$  and  $C$ . Then we know that the tails of  $S$ , the solution to  $-\Delta_x S + S = \rho$ , are at least as fat as  $\rho$ , so we can imagine that the tails of  $S \sim e^{-\alpha\langle x \rangle}$ . Then if we look at the  $M = \log(S)$  and the steady state of (5) associated to this solution, our estimates from Lemma 2.7 will give us

$$\int e^{\alpha\gamma\langle x \rangle} \rho(x) \, dx \leq C',$$

where  $\gamma < 1$  comes from Lemma 2.7 and we have  $\gamma < 1$ .

Although our discussion above shows that we would need a new moment estimate on the steady state of the linear equation in order to make such an argument work, we believe that such a fixed point argument could be carried out. Finding such an estimate is the subject of our ongoing research. In this paper, we experiment with a toy non-linear model, where we could use the estimates coming from the Foster-Lyapunov part to be able to use a fixed point argument. This gives us a better understanding of how this type of argument should work. We briefly describe our process for choosing this coupling.

The first idea was to come up with a perturbative setting to try a coupling of the form

$$-\Delta S + S = \rho_* + \eta\rho, \tag{58}$$

where  $\rho_*$  is a fixed spatial density and  $\eta$  is a small number. However, we notice that this coupling has essentially exactly the same problem with a loss of weight as the fully non-linear coupling. In order to

create a coupling we can deal with, the  $\eta\rho$  in the right hand side of (58) needs to be multiplied by a function of  $x$  that decays sufficiently fast at infinity. Therefore, we can try a coupling that looks like

$$-\Delta S + S = \rho_*(1 + \eta\rho). \quad (59)$$

Then,  $S$ , which is the solution of (59), is given by

$$S = N * (\rho_*(1 + \eta\rho)), \quad (60)$$

where  $N = \mathcal{F}^{-1}(1/(1 + |\xi|^2))$ , and  $\mathcal{F}$  represents the Fourier transform. Then, we further simplify (60) as

$$S = S_\infty(1 + \eta N * \rho)$$

where  $N$  is now a positive, smooth function and  $S_\infty$  is a smooth function. Considering this simplification allows us to keep algebra simple without losing the behaviour of (60). By this strategy we obtain the weakly non-linear, nonlocal coupling introduced in (10). Even though this weakly non-linear coupling serves as a toy model we still retain the idea of a fixed point argument on the chemoattractant profile.

Our contraction mapping argument is an adaption of what was originally an argument to show continuity of a map  $\tilde{G}$  defined on a fully non-linear coupling. In order to carry out a Schauder fixed point argument, continuity of such  $\tilde{G}$  would be needed.

Finally, the toy model we introduced, even though biologically not realistic, allows us to understand better how to use the arguments presented in this paper in the fully non-linear setting. This is a subject of ongoing work.

## Acknowledgements

The authors wish to thank Emeric Bouin, José A. Cañizo, Vincent Calvez, and Gaël Raoul for useful comments and discussions. JE was supported by FSPM postdoctoral fellowship and the grant ANR-17-CE40-0030 (both until July 2020). HY was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 865711). The authors gratefully acknowledge the support of the Hausdorff Institute for Mathematics (Bonn) through Junior Trimester Program on Kinetic Theory.

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