

On the p-adic Beilinson conjecture for number fields

Amnon Besser, Paul Buckingham, Rob de Jeu, Xavier-François Roblot

▶ **To cite this version:**

Amnon Besser, Paul Buckingham, Rob de Jeu, Xavier-François Roblot. On the p-adic Beilinson conjecture for number fields. Pure and Applied Mathematics Quarterly, 2009, 5 (1), pp.375-434. hal-00863148

HAL Id: hal-00863148
https://hal.science/hal-00863148

Submitted on 19 Sep 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE p-ADIC BEILINSON CONJECTURE FOR NUMBER FIELDS.

AMNON BESSER, PAUL BUCKINGHAM, ROB DE JEU,
AND XAVIER-FRANÇOIS ROBLOT

Dedicated to Jean-Pierre Serre on the occasion of his eightieth birthday.

Abstract. We formulate a conjectural p-adic analogue of Borel's theorem relating regulators for higher K-groups of number fields to special values of the corresponding ζ-functions, using syntomic regulators and p-adic L-functions. We also formulate a corresponding conjecture for Artin motives, and state a conjecture about the precise relation between the p-adic and classical situations. Parts of the conjectures are proved when the number field (or Artin motive) is Abelian over the rationals, and all conjectures are verified numerically in some other cases.

1. Introduction

The Beilinson conjectures about special values of L-functions [2] are a far reaching generalization of the class number formula for the Dedekind zeta function. For every algebraic variety X over the rationals it predicts the leading term of the Taylor expansions of $L(H^i(X), s)$ at certain points, up to a rational multiple, in terms of arithmetic information associated with X, namely, its algebraic K-groups $K_j(X)$ [43]. More generally, these conjectures can also be formulated for motives.

There have been several important steps taken towards verification of these conjectures in various cases, although, strictly speaking they have only been verified completely in the case where X is the spectrum of a number field, where they follow from famous theorems of Borel [10, 11].

To motivate what follows, let us briefly recall the conjecture that interests us the most (for an introduction see [23, 46]). One associates with X two cohomology groups. The first one is the Deligne cohomology $H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$, which is an $\mathbb{R}$-vector space. The second is "integral" motivic cohomology $H^i_M(X_{/\mathbb{Z}}, \mathbb{Q}(n))$, which may be defined as a certain subspace of $K_{2n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. There is a regulator map defined by Beilinson,

$$H^i_M(X_{/\mathbb{Z}}, \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))\,. \tag{1.1}$$

If $2n > i+1$ then $\det H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ has a rational structure coming from the relations between $H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$ and the de Rham and singular cohomology groups of X [46, p.30].

The first part of the conjecture is that the map in (1.1) induces an isomorphism between the lefthand side tensored with $\mathbb{R}$ and the righthand side, and consequently

2000 *Mathematics Subject Classification.* Primary: 19F27; secondary: 11G55, 11R42, 11R70.
Key words and phrases. Beilinson conjecture, Borel's theorem, number field, Artin motive, syntomic regulator, p-adic L-function.

provides a second rational structure on $\det H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$. The second part of the conjecture states that, assuming a suitable functional equation for $L(H^{i-1}(X), s)$, these two rational structures differ from each other by the leading term in the Taylor expansion of $L(H^{i-1}(X), s)$ at $s = i-n$. Because of the expected functional equation one can reformulate the conjecture in terms of $L(H^{i-1}(X), n)$ (see [2, Corollary 3.6.2] or [31, 4.12]).

As mentioned before, this conjecture has only been verified in the case of number fields, due to difficulties in the computation of motivic cohomology. What has been verified in several other cases is a form of the conjecture in which one assumes the first part. For this one finds $\dim_{\mathbb{R}} H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ elements of $H^i_M(X_{/\mathbb{Z}}, \mathbb{Q}(n))$, checks that their images under (1.1) are independent, hence should form a basis of $H^i_M(X_{/\mathbb{Z}}, \mathbb{Q}(n))$ according to the first part of the conjecture, and verifies the second part using these elements.

The idea that there should be a p-adic analogue of Beilinson's conjectures has been around since the late 80's. Such a conjecture was formulated and proved by Gros in [29, 30] in the case of Artin motives associated with Dirichlet characters. In the weak sense mentioned before, it was proved for certain CM elliptic curves in [17] (where the relation with the syntomic regulator is proved in [4] and further elucidated in [7]), and for elliptic modular forms it follows from Kato's work (see [47]).

The book [42] contains a very general conjecture about the existence and properties of p-adic L-functions, from which one can derive a p-adic Beilinson conjecture. Rather than explain this in full detail we shall give a sketch of this conjecture similar to the sketch above of the Beilinson conjecture.

For the p-adic Beilinson conjectures one has to replace Deligne cohomology with syntomic cohomology [30, 41, 3], the Beilinson regulator with the syntomic regulator, and L-functions with p-adic L-functions.

Syntomic cohomology $H^i_{\mathrm{syn}}(Y, n)$ is defined for a smooth scheme Y of finite type over a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. For the $\mathbb{Q}$-variety X we obtain, for all but finitely many primes, a map

$$H^i_M(X_{/\mathbb{Z}}, \mathbb{Q}(n)) \to H^i_{\mathrm{syn}}(Y, n) \tag{1.2}$$

where Y is a smooth model for X over $\mathbb{Z}_p$. This cohomology group is a $\mathbb{Q}_p$-vector space.

The theory of p-adic L-functions starts with Kubota and Leopoldt's p-adic ζ-function [35], $\zeta_p(s)$, which is defined by interpolating special values of complex valued Dirichlet L-functions. This principle has been extended to ζ- and L-functions in various situations, resulting in corresponding p-adic functions for totally real number fields [1, 15, 22], CM fields [32, 33] and modular forms [37]. (Given the occasion, let us note that the approach of Deligne and Ribet using modular forms was initiated by Serre [48].)

The p-adic Beilinson conjecture therefore has many similarities with its complex counterpart. However, there is a very important difference. In general, when $2n > i+1$, there is no hope that (1.2) induces an isomorphism after tensoring the lefthand side with $\mathbb{Q}_p$. To see this, consider a number field k with ring of algebraic integers $\mathcal{O}_k$. By Borel's theorem (see Theorem 2.2) we have, in accordance with Beilinson's

conjectures,

$$\dim_{\mathbb{Q}} H^1_M(\mathrm{Spec}(k)_{/\mathbb{Z}}, \mathbb{Q}(n)) = \begin{cases} r_2 \text{ when } n \geq 2 \text{ is even} \\ r_1 + r_2 \text{ when } n \geq 2 \text{ is odd} \end{cases}$$

with r_1 (resp. $2r_2$) the number of real (resp. complex) embeddings of k. Thus, motivic cohomology "knows" about the number of real and complex embeddings of k, as does, by its definition, Deligne cohomology, which in this case becomes $H^1_{\mathcal{D}}(\mathrm{Spec}(k)_{/\mathbb{R}}, \mathbb{R}(n)) \cong \{(x_\sigma)_{\sigma:k\to\mathbb{C}} \text{ in } \mathbb{R}(n-1)^{r_1+2r_2} \text{ such that } \overline{x_\sigma} = x_{\overline{\sigma}}\}$. But syntomic cohomology, which depends only on the completion of k at p, does not. In this case, we obtain $H^1_{\mathrm{syn}}(\mathrm{Spec}(\mathcal{O}_k \otimes_{\mathbb{Z}} \mathbb{Z}_p), n) \cong \mathbb{Q}_p^{r_1+2r_2}$.

The solution to this problem suggested in [42] is to make the p-adic L-function depend on a subspace of syntomic cohomology which is complementary to the image of the regulator. While this might seem artificial, there are other reasons for choosing this solution. In most cases one chooses a particular subspace and obtains a special case of the conjecture.

In the important special case of a totally real number field, or more generally an Artin motive over $\mathbb{Q}$, associated with a Galois representation whose kernel is totally real (let us call these totally real Artin motives), no such subspace is required when $n \geq 2$ is odd (see Proposition 3.12). The same holds for Artin motives where the conjugacy class of complex conjugation acts as multiplication by -1 and $n \geq 2$ is even (we may think of those as the "negative part" of CM Artin motives).

If χ denotes the character associated with either representation, then the Beilinson conjecture relates the regulator of K_{2n-1} with the Artin L-function of χ at n. For the p-adic L-functions one has to consider $\chi \otimes \omega_p^{1-n}$ with ω_p the Teichmüller character for the prime number p. Then the fixed field of the kernel of the representation underlying $\chi \otimes \omega_p^{1-n}$ is totally real, and it is perhaps no coincidence that in precisely this case the existence of a p-adic L-function that is not identicaly zero has been established, by [1, 15, 22] for the case of fields and by [28] for Artin motives.

The goal of the present work is to describe in detail the conjectures for the cases of totally real fields, totally real Artin motives, as well as the "negative part" of CM Artin motives, and describe the (conjectural) relation between the classical and p-adic conjectures. We test everything numerically in several cases, and also deduce most of the conjectures for Abelian Artin motives from work by Coleman [16].

There have been several developments that allow us to carry out this numerical verification. In [19] de Jeu proved part of Zagier's conjecture concerning a more explicit description of the K-theory (tensored with $\mathbb{Q}$) of number fields (see Section 4). While this conjecture is not known to give the K-theory of such fields in all cases, it does in practice. Thus it provides a way of computing them, and Paul Buckingham wrote a computer implementation for this. In [5] Besser and de Jeu computed the syntomic regulator for (essentially) the part of the K-theory of a number field described by Zagier's conjecture and showed that it is given by applying the p-adic polylogarithm. Those p-adic polylogarithms were invented by Coleman [16] using his theory of p-adic integration but are not so easy to compute. In [6] Besser and de Jeu devised an algorithm for this computation. Taken together, these developments allow us to compute (1.1) and (1.2) for number fields. Finally, building on earlier work in [45], Roblot has dealt with the computational aspects

of computing p-adic L-functions for Abelian characters over $\mathbb{Q}$ or a real quadratic field [44].

This paper is organized as follows. In Section 2 we recall Borel's theorem as well as various facts about L-functions and p-adic L-functions, and formulate a conjectural p-adic analogue of Borel's theorem. In Section 3 we introduce Artin motives with coefficients in a number field E in terms of representations of the Galois group, determine when the $\mathbb{Q}$-dimension of the left-hand side of (1.2) equals the $\mathbb{Q}_p$-dimension of the right-hand side (corresponding to equality in Proposition 3.12), define both classical and p-adic L-functions with coefficients in E, and formulate the motivic Beilinson conjecture with coefficients in E, Conjecture 3.18, a small part of which is the same as a conjecture by Gross. In Section 4 we describe the set-up for finding elements in the K-groups of number fields using Zagier's conjecture, and the classical and p-adic regulators on them. We also prove most of Conjecture 3.18 for Abelian Artin motives over $\mathbb{Q}$ (Proposition 4.17 and Remark 4.18). In Section 5 we discuss a few computational aspects of implementing Zagier's conjecture and describe the Artin motives that we consider later for the numerical examples, and in the process prove Gross's conjecture for Artin motives obtained from S_3 and D_8-extensions of $\mathbb{Q}$. Then in Section 6 we sketch how the p-adic L-functions can be computed in certain cases, and make the required Brauer induction explicit for the Artin motives we want to consider. Finally, the last section is devoted to the results of the numerical calculations for examples.

Amnon Besser, Paul Buckingham and Rob de Jeu would like to thank the EC network Arithmetic Algebraic Geometry for travel support. Rob de Jeu would like to thank the Tata Institute of Fundamental Research for a productive stay during which this paper was worked on, and the Nuffield Foundation for a grant under the Undergraduate Research Bursary programme (NUF-URB03) that enabled Paul Buckingham to develop a computer program for finding the necessary elements in K-theory. Amnon Besser would like to thank the Institute for Advanced Study in Princeton for a stay during which part of the paper was worked on, and the Bell companies Fellowship and the James D. Wolfensohn fund for financial support while at the institute. Finally, the authors would like to thank Alfred Weiss for very useful explanations about p-adic L-functions, and, in particular, for bringing to their attention the paper [28].

Notation

Throughout the paper, for an Abelian group A we let $A_{\mathbb{Q}}$ denote $A \otimes_{\mathbb{Z}} \mathbb{Q}$. If B is a subgroup of $\mathbb{C}$ and n an integer then we let $B(n) = (2\pi i)^n B \subseteq \mathbb{C}$ We normalize the absolute value $|\cdot|_p$ on the field of p-adic numbers $\mathbb{Q}_p$ in such a way that $|p|_p = p^{-1}$, and use the same notation for its extensions to an algebraic closure $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$.

2. The p-adic Beilinson conjecture for totally real fields

Let k be a number field with r_1 real embeddings, $2r_2$ complex embeddings, ring of algebraic integers $\mathcal{O}_k$, and discriminant D_k. As is well-known, $\mathcal{O}_k^*$ is a finitely generated Abelian group of rank $r = r_1 + r_2 - 1$, and its regulator R satisfies $w\sqrt{|D_k|}\,\mathrm{Res}_{s=1}\zeta_k(s) = 2^{r_1}(2\pi)^{r_2}\,|\mathrm{Cl}(\mathcal{O}_k)|\,R$, with $\mathrm{Cl}(\mathcal{O}_k)$ the class group of $\mathcal{O}_k$, and w the number of roots of unity in k. Because $K_0(\mathcal{O}_k) \cong \mathrm{Cl}(\mathcal{O}_k) \oplus \mathbb{Z}$ and $K_1(\mathcal{O}_k) \cong \mathcal{O}_k^*$, so $|\mathrm{Cl}(\mathcal{O}_k)| = |K_0(\mathcal{O}_k)_{\mathrm{tor}}|$ and $w = |K_1(\mathcal{O}_k)_{\mathrm{tor}}|$, this can be interpreted as a statement about the K-theory of $\mathcal{O}_k$, and it is from this point

of view that it can be generalized to $\zeta_k(n)$ for $n \geq 2$. Namely, in [43], Quillen proved that $K_m(\mathcal{O}_k)$ is a finitely generated Abelian group for all m. Borel in [10] computed its rank when $m \geq 2$. For m even this rank is zero, but for odd m it is $r_1 + r_2$ or r_2, and in [11] he showed that a suitably defined regulator of $K_{2n-1}(\mathcal{O}_k)$ is related to $\zeta_k(n)$. Since $K_{2n-1}(\mathcal{O}_k) \xrightarrow{\sim} K_{2n-1}(k)$ when $n \geq 2$ we can rephrase his results for $K_{2n-1}(\mathcal{O}_k)$ in terms of $K_{2n-1}(k)$. Also, we replace Borel's regulator map $\mathrm{reg}_B : K_{2n-1}(\mathbb{C}) \to \mathbb{R}(n-1)$ $(n \geq 2)$ with Beilinson's regulator map reg_∞ (see [46, § 4]), which is half the Borel map regulator by [12, Theorem 10.9]. Because $k \otimes_{\mathbb{Q}} \mathbb{C} \cong \oplus_{\sigma : k \to \mathbb{C}} \mathbb{C}$, and $n > \dim \mathrm{Spec}(k \otimes \mathbb{C})$, we obtain by [46, p.9]

$$\begin{aligned} H^1_{\mathcal{D}}(\mathrm{Spec}(k \otimes \mathbb{C})_{/\mathbb{R}}, \mathbb{R}(n)) &\cong H^0(\mathrm{Spec}(k \otimes \mathbb{C})_{/\mathbb{R}}, \mathbb{R}(n-1)) \\ &\cong \Big(\bigoplus_{\sigma : k \to \mathbb{C}} \mathbb{R}(n-1) \Big)^+, \end{aligned} \tag{2.1}$$

which consists of those $(x_\sigma)_\sigma$ with $x_{\overline{\sigma}} = \overline{x_\sigma}$. Finally, for any embedding $\sigma : k \to \mathbb{C}$ we let $\sigma_* : K_{2n-1}(k) \to K_{2n-1}(\mathbb{C})$ be the induced map.

Theorem 2.2. (Borel) *Let k be a number field of degree d, with r_1 real embeddings and $2r_2$ complex embeddings, and let $n \geq 2$. Then the rank m_n of $K_{2n-1}(k)$ equals r_2 if n is even and $r_1 + r_2$ if n is odd. Moreover, the map*

$$\begin{aligned} K_{2n-1}(k) &\to \bigoplus_{\sigma : k \to \mathbb{C}} \mathbb{R}(n-1) \\ \alpha &\mapsto (\mathrm{reg}_\infty \circ \sigma_*(\alpha))_\sigma \end{aligned} \tag{2.3}$$

embeds $K_{2n-1}(k)/\mathrm{torsion}$ as a lattice in $(\oplus_\sigma \mathbb{R}(n-1))^+ \cong \mathbb{R}^{m_n}$, and the volume $V_n(k)$ of a fundamental domain of this lattice satisfies

$$\zeta_k(n) \sqrt{|D_k|} = q \pi^{n(d-m_n)} V_n(k) \tag{2.4}$$

for some q in $\mathbb{Q}^*$.

Remark 2.5. In Theorem 2.2 $m_n = 0$ precisely when k is totally real and $n \geq 2$ is odd. In this case the given relation holds (with $V_n(k) = 1$) by the Siegel-Klingen theorem [40, Chapter VII, Corollary 9.9].

This theorem is equivalent with Beilinson's conjecture for k. In order to deal with this in detail (see Remarks 2.20 and 3.24) and in order to introduce p-adic L-functions we recall some facts about Artin L-functions [40, Chapter VII, § 10-12]. Let k be a number field, $d = [k : \mathbb{Q}]$, and let χ be a $\mathbb{C}$-valued Artin character of $\mathrm{Gal}(\overline{k}/k)$. For a prime number ℓ we define

$$\mathrm{Eul}_\ell(s, \chi, k) = \prod_{\mathfrak{l} | \ell} \mathrm{Eul}_{\mathfrak{l}}(s, \chi, k)$$

where $\mathrm{Eul}_{\mathfrak{l}}(s, \chi, k)$ is the reciprocal of the Euler factor for $\mathfrak{l}$ and the product is over primes $\mathfrak{l}$ of k lying above ℓ. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we can write the Artin L-function of χ as

$$L(s, \chi, k) = \prod_{\ell} \mathrm{Eul}_\ell(s, \chi, k)^{-1}.$$

For an infinite place v of k we let

$$L_v(s, \chi, k) = \begin{cases} L_{\mathbb{C}}(s)^{\chi(1)} & \text{when } v \text{ is complex,} \\ L_{\mathbb{R}}(s)^{n^+} L_{\mathbb{R}}(s+1)^{n^-} & \text{when } v \text{ is real,} \end{cases}$$

where $L_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $L_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $n^{\pm} = \frac{d}{2}\overline{\chi}(1) \pm \sum_v \frac{1}{2}\overline{\chi}(\mathrm{Fr}_w)$, with the sum taken over the real places v of k and Fr_w the generator of the image of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ in $\mathrm{Gal}(\overline{k}/k)$ corresponding to any extension $w : \overline{k} \to \mathbb{C}$ of v. Then

$$\Lambda(s,\chi,k) = L(s,\chi,k)\prod_{v|\infty} L_v(s,\chi,k)$$

extends to a meromorphic function, satisfying the functional equation (cf. [40, p. 541])

$$\Lambda(s,\chi,k) = W(\chi)C(\chi,k)^{1/2}C(\chi,k)^{-s}\Lambda(1-s,\overline{\chi},k)$$

with $W(\chi)$ a constant of absolute value 1 and $C(\chi,k) = |D_k|^{\chi(1)}\mathrm{Nm}_{k/\mathbb{Q}}(\mathfrak{f}(\chi,k))$ for $\mathfrak{f}(\chi,k)$ the Artin conductor of χ. Therefore

$$\begin{aligned} L(1-s,\overline{\chi},k) = W(\overline{\chi})C(\overline{\chi},k)^{s-\frac{1}{2}}(2(2\pi)^{-s}\Gamma(s))^{d\overline{\chi}(1)} \\ \times (\cos(\pi s/2))^{n^+}(\sin(\pi s/2))^{n^-}L(s,\chi,k)\,. \end{aligned} \tag{2.6}$$

Following [28, pp. 80–81] we shall now describe a p-adic L-function $L_p(s,\chi,k)$ when k is a totally real number field, p a prime and $\chi : \mathrm{Gal}(\overline{k}/k) \to \overline{\mathbb{Q}}_p$ a suitable Artin character. We begin with the case of 1-dimensional Artin characters.

If $\sigma : \overline{\mathbb{Q}}_p \to \mathbb{C}$ is any isomorphism then $\sigma\circ\chi$ is a complex Artin character, so we have the Artin L-function $L(s,\sigma\circ\chi,k)$. We may also view χ as a character of a suitable ray class group, so by [40, Corollary 9.9 and page 509] all $L(m,\sigma\circ\chi,k)$ for $m \in \mathbb{Z}_{\leq 0}$ are in $\mathbb{Q}(\sigma\circ\chi)$ and the values

$$L^*(m,\chi,k) = \sigma^{-1}(L(m,\sigma\circ\chi,k)) \tag{2.7}$$

are independent of the choice of σ.

In the same way, we define

$$\mathrm{Eul}^*_\ell(m,\chi,k) = \sigma^{-1}(\mathrm{Eul}_\ell(m,\sigma\circ\chi,k))\,, \tag{2.8}$$

which is clearly independent of the choice of σ. To construct the p-adic L-function one finds a p-adic analytic or meromorphic function on an open ball around 0 that interpolates the values $L^*(m,\chi,k)$. For 1-dimensional χ with the fixed field k_χ of its kernel totally real this was achieved independently by Deligne and Ribet [22], Barsky [1], and Cassou-Noguès [14, 15]. We shall sketch a proof of the following theorem and Remark 2.13 below in Section 6. (When k_χ is not totally real the p-adic L-function is identically zero since the values interpolated are all zero by the functional equation of the L-function.)

Theorem 2.9. *For p prime, let $\mathcal{B}$ in $\mathbb{C}_p$ be the open ball with centre 0 and radius $qp^{-1/(p-1)}$ where $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. If k is a totally real number field and $\chi : \mathrm{Gal}(\overline{k}/k) \to \overline{\mathbb{Q}}_p$ a 1-dimensional Artin character, then there exists a unique $\mathbb{C}_p$-valued function $L_p(s,\chi,k)$ on $\mathcal{B}$ satisfying the following properties:*

1. *$L_p(s,\chi,k)$ is analytic if χ is non-trivial and meromorphic with at most a simple pole at $s = 1$ if χ is trival;*
2. *if m is a negative integer such that $m \equiv 1$ modulo $\varphi(q)$ then*

$$L_p(m,\chi,k) = \mathrm{Eul}^*_p(m,\chi,k)L^*(m,\chi,k)\,.$$

If $\chi : \mathrm{Gal}(\overline{k}/k) \to \overline{\mathbb{Q}}_p$ is any Artin character then by Brauer's induction theorem [40, (10.3)] there exist 1-dimensional Artin characters $\chi_1, \ldots, \chi_t$ on subgroups $G_1, \ldots, G_t$ of $\mathrm{Gal}(\overline{k}/k)$ of finite index, and integers $a_1, \ldots, a_t$, such that

$$\chi = \sum_{i=1}^{t} a_i \mathrm{Ind}_{G_i}^{\mathrm{Gal}(\overline{k}/k)}(\chi_i)\,. \tag{2.10}$$

If k_χ is totally real then we can assume that the same holds for the fixed fields k_i of the G_i, and we define the p-adic L-function of χ by

$$L_p(s, \chi, k) = \prod_{i=1}^{t} L_p(s, \chi_i, k_i)^{a_i}\,, \tag{2.11}$$

which is a meromorphic function on $\mathcal{B}$ (see Section 6). For any negative integer $m \equiv 1$ modulo $\varphi(q)$ the value $L_p(m, \chi, k)$ is defined and equals $\mathrm{Eul}_p^*(m, \chi, k)L^*(m, \chi, k)$ by well-known properties of Artin L-functions (see [40, Prop. 10.4(iv)]), showing the function is independent of how we express χ as a sum of induced 1-dimensional characters.

Remark 2.12. In [28], Greenberg proves that the Main Conjecture of Iwasawa theory implies the p-adic Artin conjecture, that is, that the p-adic L-function of an Artin character χ is analytic on the open ball $\mathcal{B}$ if it does not contain the trivial character, and has at most a simple pole at $s = 1$ otherwise. It therefore follows from the proof of the Main Conjecture by Mazur and Wiles [38] for p odd that $L_p(n, \chi, k)$ for $p \neq 2$ is defined for all integers $n \neq 1$. In particular, the values of the p-adic L-functions in Conjecture 2.17 below exist by Theorem 2.9, and those in Conjecture 3.18 exist when $p \neq 2$ but have to be assumed to exist when $p = 2$.

Remark 2.13. Let $W_p \subset \mathbb{Z}_p^*$ be the subgroup of $(p-1)$-th roots of unity if p is odd and let $W_2 = \{\pm 1\}$. The Teichmüller character on $\mathrm{Gal}(\overline{k}/k)$ is defined as the composition

$$\omega_p : \mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/q\mathbb{Z})^* \xrightarrow{\sim} W_p\,, \tag{2.14}$$

where the last map sends an element of $(\mathbb{Z}/q\mathbb{Z})^*$ to the unique element of W_p to which it is congruent modulo q. For an Artin character $\chi : \mathrm{Gal}(\overline{k}/k) \to \overline{\mathbb{Q}}_p$ and an integer l, $\chi\omega_p^l$ is also an Artin character. If $m \leq 0$ satisfies $l + m \equiv 1$ modulo $\varphi(q)$ and either χ is 1-dimensional or k_χ is totally real then $L_p(m, \chi\omega_p^l, k) = \mathrm{Eul}_p^*(m, \chi, k)L^*(m, \chi, k)$.

The p-adic Beilinson conjecture is going to predict the special values of p-adic L-functions in terms of a p-adic regulator. The required regulator is the syntomic regulator [3, Theorem 7.5]. Let F be a complete discretely valued field of characteristic 0 with perfect residue field of characteristic p and let X be a scheme that is smooth and of finite type over the valuation ring $\mathcal{O}_F$. Then the above mentioned paper associates to X its rigid syntomic cohomologies $H^i_{\mathrm{syn}}(X, n)$, as well as syntomic regulators (i.e., Chern characters) $\mathrm{reg}_p : K_{2n-i}(X) \to H^i_{\mathrm{syn}}(X, n)$. In this work, unlike in [3], we shall need to change the base field F. We therefore prefer to denote the syntomic cohomology by $H^i_{\mathrm{syn}}(X/\mathcal{O}_F, n)$. For the formulation of the conjecture the following basic fact is required.

Lemma 2.15. *We have $H^1_{\mathrm{syn}}(\mathrm{Spec}(\mathcal{O}_F)/\mathcal{O}_F, n) \cong F$, for all $n > 0$ and consequently we have a syntomic regulator $\mathrm{reg}_p : K_{2n-1}(\mathcal{O}_F) \to F$. Furthermore, the*

map reg_p *commutes in the obvious way with finite extensions of fields and with automorphisms of such fields, provided that their residue fields are algebraic over the prime field.*

Proof. The first claim follows from part 3 of [3, Proposition 8.6]. For the second claim we note that by part 4 of the same proposition we have in this case an isomorphism between syntomic and modified syntomic cohomology (the latter only exists under the additional assumption on the residue field). The compatibility with finite base changes now follows from the same result for modified syntomic cohomology in [3, Proposition 8.8]. The same holds for automorphisms, although not explicitly stated in the above reference, since the relevant base change results, e.g., Proposition 8.6.4, hold for this type of base change as well. □

As a consequence of the lemma we can also, by abuse of notation, define for any complete discretely valued subfield $F \subset \overline{\mathbb{Q}}_p$ the regulator $\mathrm{reg}_p : K_{2n-1}(\mathcal{O}_F) \to F$. In [5], two of the authors of the present work showed how one can sometimes compute the map reg_p by using p-adic polylogarithms.

We now restrict our attention to a totally real number field k with $[k : \mathbb{Q}] = d$. Our goal will be to formulate a conjecture that is the p-adic analogue of Theorem 2.2 for $K_{2n-1}(k)$ with $n \geq 2$. Since this K-group is torsion when n is even but has rank d when n is odd, we only consider odd $n \geq 2$.

In preparation for the more general construction that will follow in Section 3 let us first reformulate Theorem 2.2 in this special case. Let $a_1, \dots, a_d$ form a $\mathbb{Z}$-basis of $\mathcal{O}_k$ and let $\sigma_1^\infty, \dots, \sigma_d^\infty$ be the embeddings of k into $\mathbb{C}$. Then we define $D_k^{1/2,\infty} = \det(\sigma_i^\infty(a_j))$, a real root of the discriminant of k. Similarly, if $\alpha_1, \dots, \alpha_d$ form a $\mathbb{Z}$-basis of $K_{2n-1}(k)/\text{torsion}$ then we let $R_{n,\infty}(k) = \det(\mathrm{reg}_\infty \circ \sigma_{i*}^\infty(\alpha_j))$. Then $D_k^{1/2,\infty}$ and $R_{n,\infty}(k)$ are well-defined up to sign, and the relation in Theorem 2.2 in this case is equivalent with

$$\zeta_k(n) D_k^{1/2,\infty} = q(n,k) R_{n,\infty}(k) \tag{2.16}$$

with $q(n,k)$ in $\mathbb{Q}^*$.

Now let $F \subset \overline{\mathbb{Q}}_p$ be the topological closure of the Galois closure of k embedded in $\overline{\mathbb{Q}}_p$ in any way. If $\sigma_1^p, \dots, \sigma_d^p$ are the embeddings of k into F then $D_k^{1/2,p} = \det(\sigma_i^p(a_j))$ is a root in F of the discriminant of k. For $\sigma : k \to F$ an embedding we denote the induced map $K_{2n-1}(k) \to K_{2n-1}(F)$ by σ_*. Then we define a p-adic regulator in F by $R_{n,p}(k) = \det(\mathrm{reg}_p \circ \sigma_{i*}^p(\alpha_j))$. Both $D_k^{1/2,p}$ and $R_{n,p}(k)$ are well-defined up to sign.

Remark 2.13 suggests that the role of $\zeta_k(n)$ in a p-adic analogue of Theorem 2.2 should be played by $L_p(n, \omega_p^{1-n}, k)/\mathrm{Eul}_p(n,k)$ where $\zeta_k(s) = \prod_l \mathrm{Eul}_l(s,k)^{-1}$ for $\mathrm{Re}(s) > 1$, so we can hope that

$$L_p(n, \omega_p^{1-n}, k) D_k^{1/2,p} = q_p(n,k) \,\mathrm{Eul}_p(n,k) R_{n,p}(k)$$

for some $q_p(n,k)$ in $\mathbb{Q}^*$.

More precisely, because $R_{n,\infty}(k)/D_k^{1/2,\infty}$ and $R_{n,p}(k)/D_k^{1/2,p}$ are invariant under reordering the σ_i^∞ or σ_i^p, and transform in the same way if we change the bases of $\mathcal{O}_k$ and $K_{2n-1}(k)/\text{torsion}$, we can make the following conjecture.

Conjecture 2.17. *For k a totally real number field, p prime, and $n \geq 2$ odd, we have, with notation as above:*

(1) *in F the equality*

$$L_p(n,\omega_p^{1-n},k)D_k^{1/2,p} = q_p(n,k)\,\mathrm{Eul}_p(n,k)R_{n,p}(k)$$

holds for some $q_p(n,k)$ in $\mathbb{Q}^*$;
(2) *in fact, $q_p(n,k) = q(n,k)$;*
(3) *$L_p(n,\omega_p^{1-n},k)$ and $R_{n,p}(k)$ are non-zero.*

As mentioned in the introduction, this, and the corresponding parts of Conjecture 3.18 below, can be deduced (with some effort) from a much more general conjecture of Perrin-Riou [42, 4.2.2].

Remark 2.18. The conjecture is similar to the result that the residue of $\zeta_p(s,k)$ at $s=1$ is related to the Leopoldt regulator of $\mathcal{O}_k^*$ through exactly the same formula as for the residue of $\zeta(s,k)$ and the Dirichlet regulator [18], with part(4) corresponding to the Leopoldt conjecture. However, we have not tried to determine if Colmez's normalization of the sign for the regulator is the same as here, especially given the sign error in the proof of Lemma 4.3 in loc. cit. (see Section 6).

Remark 2.19. We can use a basis of a subgroup of finite index of $K_{2n-1}(k)/\text{torsion}$ in the definition of $R_{n,\infty}(k)$ and $R_{n,p}(k)$, or even a $\mathbb{Q}$-basis of $K_{2n-1}(k)_{\mathbb{Q}}$, without affecting the rationality of $q(n,k)$, $q_p(n,k)$ or their equality. Similarly we can replace the $\mathbb{Z}$-basis of $\mathcal{O}_k$ with a $\mathbb{Q}$-basis of k in the definitions of $D_k^{1/2,\infty}$ and $D_k^{1/2,p}$.

Remark 2.20. As is well-known, for k and n as in Conjecture 2.17, by (2.6) $\zeta_k(s)$ at $s=1-n$ has a zero of order d and the first non-zero coefficient in its Taylor expansion, $\zeta_k^\sharp(1-n)$, equals $(2\pi i)^{d(1-n)}((n-1)!/2)^d D_k^{n-1/2}\zeta_k(n)$. If we take $D_k^{1/2,\infty} = D_k^{1/2}$ in (2.16) then we obtain

$$\zeta_k^\sharp(1-n) = ((n-1)!/2)^d q(n,k) D_k^{n-1}\widetilde{R}_{n,\infty}(k)$$

for Beilinson's renormalized regulator $\widetilde{R}_{n,\infty}(k) = (2\pi i)^{d(1-n)}R_{n,\infty}(k)$. In computer calculations $((n-1)!/2)^d q(n,k)D_k^{n-1}$ often has only relatively small prime factors, so the larger prime factors in $q(n,k)$ correspond to D_k^{1-n}. This phenomenon also occurs in the calculations for Conjecture 3.18 below (see Remark 7.8).

Remark 2.21. (1) The thought that $L_p(n,\omega_p^{1-n},k)$ is non-zero for $n\geq 2$ and odd when k is a totally real Abelian extension of $\mathbb{Q}$ is mentioned by C. Soulé in [50, 3.4].

(2) F. Calegari [13] (see also [8]) proved that, for $p=2$ and 3, $\zeta_p(3)$, which in those cases equals $L_p(3,\omega_p^{-2},\mathbb{Q})$, is irrational. (More results along these lines are described in Remark 3.20(3).)

(3) Parts (1) and (2) of Conjecture 2.17 hold when k is a totally real Abelian number field, and in fact the corresponding parts of a much stronger conjecture that we shall describe in Section 3 hold for cyclotomic fields (see Proposition 4.17).

(4) We numerically verified part (3) of Conjecture 2.17, and its more refined version Conjecture 3.18(4) below, in certain cases; see Remark 4.19 and Section 7.

3. A motivic version of the conjecture

If E is any extension of $\mathbb{Q}$, and $k/\mathbb{Q}$ is finite and Galois with Galois group G, then we let $M^E = E\otimes_{\mathbb{Q}} k$ and $K_{2n-1}(k)_E = E\otimes_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}}$, which are $E[G]$-modules. The goal of this section is to refine Conjecture 2.17 to a conjecture for

Artin motives with coefficients in E, or equivalently, idempotents in the group ring $E[G]$, when E is a number field.

Definition 3.1. For an idempotent π in $E[G]$ we let $M_\pi^E = \pi M^E$ and $K_{2n-1}(M_\pi^E) = \pi K_{2n-1}(k)_E$.

Now fix an embedding $\phi_\infty : k \to \mathbb{C}$. The pairing $G \times k \to \mathbb{C}$ mapping (σ, a) to $\phi_\infty(\sigma(a))$ leads to an E-bilinear pairing

$$\begin{aligned} E[G] \times M^E &\to E \otimes_{\mathbb{Q}} \mathbb{C} \\ (e\sigma, e' \otimes a) &\mapsto ee' \otimes \phi_\infty(\sigma(a))\,. \end{aligned} \tag{3.2}$$

and we consider its restriction

$$(\,\cdot\,,\,\cdot\,)_\infty : E[G]\pi \times M_\pi^E \to E \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

Similarly, replacing ϕ_∞ with a fixed embedding $\phi_p : k \to \overline{\mathbb{Q}}_p$ we obtain

$$(\,\cdot\,,\,\cdot\,)_p : E[G]\pi \times M_\pi^E \to E \otimes_{\mathbb{Q}} F\,.$$

where $F \subset \overline{\mathbb{Q}}_p$ is the topological closure of $\phi_p(k)$, which is independent of ϕ_p since $k/\mathbb{Q}$ is Galois.

Lemma 3.3. *Let π in $E[G]$ be an idempotent. Then* $\dim_E(M_\pi^E) = \dim_E(E[G]\pi)$.

Proof. Since $\pi^2 = \pi$, (3.2) is identically 0 on $E[G]\pi \times M_{1-\pi}^E$ and $E[G](1-\pi) \times M_\pi^E$. But the determinant of this pairing is, up to multiplication by E^*, equal to $D_k^{1/2,\infty}$, hence non-zero. This implies the lemma. □

We now introduce pairings similar to $(\cdot,\cdot)_\infty$ and $(\cdot,\cdot)_p$ but replacing M_π^E with $K_{2n-1}(M_\pi^E)$. If we denote the map $K_{2n-1}(k)_{\mathbb{Q}} \to K_{2n-1}(\mathbb{C})_{\mathbb{Q}}$ induced by ϕ_∞ by $\phi_{\infty *}$ and let Φ_∞ be the composition with the Beilinson regulator map reg_∞ for $\mathbb{C}$,

$$\Phi_\infty = \mathrm{reg}_\infty \circ \phi_{\infty *} : K_{2n-1}(k)_{\mathbb{Q}} \to K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \to \mathbb{R}(n-1) \subset \mathbb{C}\,,$$

then the pairing $G \times K_{2n-1}(k)_{\mathbb{Q}} \to \mathbb{C}$ given by mapping (σ, α) to $\Phi_\infty(\sigma(\alpha))$ gives rise to an E-bilinear pairing

$$\begin{aligned} E[G] \times K_{2n-1}(M^E) &\to E \otimes_{\mathbb{Q}} \mathbb{C} \\ (e\sigma, e' \otimes \alpha) &\mapsto ee' \otimes \Phi_\infty(\sigma(\alpha)) \end{aligned} \tag{3.4}$$

and we consider its restriction

$$[\,\cdot\,,\,\cdot\,]_\infty : E[G]\pi \times K_{2n-1}(M_\pi^E) \to E \otimes_{\mathbb{Q}} \mathbb{C}\,.$$

By Lemma 2.15 and the definition of F the syntomic regulator gives us

$$\mathrm{reg}_{\mathrm{syn}} : K_{2n-1}(\mathcal{O}_F)_{\mathbb{Q}} \to F\,.$$

If we write ϕ_{p*} for the composition $K_{2n-1}(k)_{\mathbb{Q}} \cong K_{2n-1}(\mathcal{O}_k)_{\mathbb{Q}} \to K_{2n-1}(\mathcal{O}_F)_{\mathbb{Q}}$, with the second map induced by $\phi_p : k \to F$, and let Φ_p be the composition

$$\mathrm{reg}_{\mathrm{syn}} \circ \phi_{p*} : K_{2n-1}(k)_{\mathbb{Q}} \to K_{2n-1}(F)_{\mathbb{Q}} \to F\,,$$

then we can similarly obtain a pairing

$$[\,\cdot\,,\,\cdot\,]_p : E[G]\pi \times K_{2n-1}(M_\pi^E) \to E \otimes_{\mathbb{Q}} F\,.$$

We now fix ordered E-bases of $E[G]\pi$, M_π^E, and $K_{2n-1}(M_\pi^E)$.

Definition 3.5. For $* = \infty$ or p we let $D(M_\pi^E)^{1/2,*}$ be the determinant of the pairing $(\cdot,\cdot)_*$, computed with respect to our fixed bases of $E[G]\pi$ and M_π^E.

Note that $D(M_\pi^E)^{1/2,\infty}$ is non-zero by the proof of Lemma 3.3, and the same argument works for $D(M_\pi^E)^{1/2,p}$.

Definition 3.6. If $\dim_E(E[G]\pi)$ equals $\dim(K_{2n-1}(M_\pi^E))$ then for $* = \infty$ or p we let $R_{n,\infty}(M_\pi^E)$ be the determinant of the pairing $[\cdot,\cdot]_*$, computed with respect to our fixed bases of $E[G]\pi$ and $K_{2n-1}(M_\pi^E)$.

For future use we prove the following.

Lemma 3.7. *If* $\dim_E(E[G]\pi) = \dim(K_{2n-1}(M_\pi^E))$ *then*

1. $R_{n,\infty}(M_\pi^E)/D(M_\pi^E)^{1/2,\infty}$ *is independent of the basis of* $E[G]\pi$*, of* ϕ_∞*, and lies in* $E \otimes \mathbb{R}$*;*
2. $R_{n,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$ *is independent of the basis of* $E[G]\pi$*, of* ϕ_p*, and lies in* $E \otimes \mathbb{Q}_p$*.*

Proof. We prove the second statement, the proof of the first being entirely similar. Choosing a different E-basis of $E[G]\pi$ corresponds to letting an E-linear transformation act on $E[G]\pi$, and in the given quotient the resulting determinant cancels. Since $k/\mathbb{Q}$ is Galois we get all possible embeddings of k into $\overline{\mathbb{Q}}_p$ by replacing ϕ_p with $\phi_p \circ \sigma$ for σ in G. For both $R_{n,p}(M^E)$ and $D^{1/2,p}(M^E)$ this corresponds to letting σ act on $E[G]\pi$, and the resulting determinant cancels as before. That the quotient lies in $E \otimes \mathbb{Q}_p$ follows because it lies in $E \otimes F$ and we have just proved that it is invariant under $\mathrm{Gal}(F/\mathbb{Q}_p)$ by Lemma 2.15. □

We now investigate when the two dimensions in Definition 3.6 are equal. The answer is given by Proposition 3.12 below, but we need some preliminary results.

Proposition 3.8. *If* $k/\mathbb{Q}$ *is a finite Galois extension with Galois group* G *and* π *is an idempotent in* $E[G]$ *then* $\dim_E(K_{2n-1}(M_\pi^E)) \le \dim_E(E[G]\pi)$*. Equality holds precisely when any (hence every)* τ *in the image in* G *of the conjugacy class of complex conjugation in* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts by* $(-1)^{n-1}$ *on* $E[G]\pi$*.*

Proof. For the statement we can first replace E by a finitely generated subfield since π contains only finitely many elements of E, next embed E into $\mathbb{C}$ and use this embedding to enlarge E to $\mathbb{C}$. So we may assume that $E = \mathbb{C}$.

According to Theorem 2.2 the pairing (3.4) gives an injection

$$\begin{aligned} \mathbb{R} \otimes_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}} &\to \mathbb{R}[G]^\vee = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[G], \mathbb{R}) \\ \alpha &\mapsto f_\alpha \text{ with } f_\alpha(\sigma) = \frac{1}{(2\pi i)^{n-1}} \Phi_\infty(\sigma(\alpha)) \end{aligned} \tag{3.9}$$

and by extending the coefficients we get an injection $\mathbb{C} \otimes_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}} \to \mathbb{C}[G]^\vee$. The image of $\pi(K_{2n-1}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C})$ under the last map vanishes on $\mathbb{C}[G](1-\pi)$ since $(1-\pi)\pi = 0$ so that $\pi K_{2n-1}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ injects into $(\mathbb{C}[G]/\mathbb{C}[G](1-\pi))^\vee \cong (\mathbb{C}[G]\pi)^\vee$, which proves our first inequality.

As for equality, we know by Theorem 2.2 that (3.9) has as its image the subspace of $\mathbb{R}[G]^\vee$ where, under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathbb{R}[G]^\vee$ via $(\tilde{\sigma} f)(\sigma) = f(\tilde{\sigma}^{-1}\sigma)$, the conjucagy class of complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts as multiplication by $(-1)^{n-1}$. The same will therefore hold with complex coefficients. Since any element τ in the conjugacy class of complex conjugation has order 1 or 2, $\mathbb{C}[G]\pi$ decomposes into eigenspaces for the eigenvalues ± 1 and the desired equality can only hold if τ acts as multiplication by $(-1)^{n-1}$ on all of $\mathbb{C}[G]\pi$. □

We now determine precisely when the equality of dimensions as in Proposition 3.8 can occur, and for this we need a preliminary result.

Proposition 3.10. *Let ψ be a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\overline{\mathbb{Q}}$ that factorizes through the Galois group of a finite Galois extension of $\mathbb{Q}$ and for which $\psi(\tau)$ acts as multiplication by $(-1)^{n-1}$ for any τ in the conjugacy class of complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the fixed field of $\mathrm{Ker}(\psi)$ is a finite Galois extension of $\mathbb{Q}$ that is totally real if n is odd, and CM if n is even.*

Proof. That the fixed field k of $\mathrm{Ker}(\psi)$ is a finite Galois extension of $\mathbb{Q}$ is clear. When n is odd $\psi(\tau) = 1$ for any τ in the conjugacy class of complex conjugation so that k is totally real. For even n we let ω be the composition $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\mu_4)/\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\} \subset \overline{\mathbb{Q}}^*$. Then $\mathrm{Ker}(\psi\omega)$ contains τ and $\mathrm{Ker}(\psi) \cap \mathrm{Ker}(\omega)$, hence its fixed field is a totally real Galois extension k^+ of $\mathbb{Q}$, contained in $k(\mu_4)$. Similarly k is contained in the CM field $k^+(\mu_4)$ so, since k is not totally real, it must be CM. □

Remark 3.11. For a CM field k with $k/\mathbb{Q}$ Galois, its maximal totally real subfield is Galois over $\mathbb{Q}$ and is the fixed field of an element of order two in the centre of $\mathrm{Gal}(k/\mathbb{Q})$, which we shall refer to as the complex conjugation of k.

Proposition 3.12. *Let $k/\mathbb{Q}$ be a finite Galois extension with Galois group G, E any extension of $\mathbb{Q}$, and π an idempotent of $E[G]$. Let k' be the fixed field of the kernel of the representation of G on $E[G]\pi$. Then for $n \geq 2$ the equality $\dim_E(E[G]\pi) = \dim_E(K_{2n-1}(M_\pi^E))$ holds precisely in the following cases:*

1. *k' is totally real and n is odd;*
2. *k' is a CM field, n is even, and the complex conjugation of k' acts on $E[G]\pi$ as multiplication by -1.*

Proof. From Propositions 3.8 and 3.10 we see that there cannot be any other cases. Conversely, by Proposition 3.8 equality holds in both. □

We now recall and introduce some terminology for later use.

Definition 3.13. Let G be a finite group and E any extension of $\mathbb{Q}$.

1. If π is a central idempotent of $E[G]$ such that $E[G]\pi$ is a minimal (non-zero) 2-sided ideal of $E[G]$ then π is a primitive central idempotent.
2. If π is a primitive central idempotent of $E[G]$ such that $E[G]\pi \cong M_m(E)$ as E-algebras for some m then we call a primitive idempotent corresponding to π any element in $E[G]\pi \subseteq E[G]$ that maps to a matrix in $M_m(E)$ that is conjugate to a matrix with 1 in the upper left corner and 0's elsewhere. (Since all automorphisms of $M_m(E)$ as E-algebra are inner this is independent of the isomorphism $E[G]\pi \cong M_m(E)$.)

Remark 3.14. (1) If $E[G]\pi \cong M_m(E)$ then any idempotent in $E[G]\pi$ can be written as a sum of orthogonal primitive idempotents corresponding to π as one sees immediately by diagonalizing the matrix A that π maps to, which satisfies $A^2 = A$.

(2) If $G = \mathrm{Gal}(k/\mathbb{Q})$ and π is a primitive central idempotent of $E[G]$ then the dimensions of $E[G]\tilde{\pi}$ and $\tilde{\pi}K_{2n-1}(k)_E$ are equal for some non-zero idempotent $\tilde{\pi}$ in $E[G]\pi$ if and only if the same holds for one (hence any) primitive idempotent corresponding to π. Indeed, the dimensions for π do not change if we replace it with a conjugate in $E[G]$, and they add up for sums of orthogonal idempotents.

We now introduce L-functions, both classical and p-adic, in the following context. Let E be a finite extension of $\mathbb{Q}$, k a number field, and consider a character ψ of a representation of $G = \mathrm{Gal}(\overline{k}/k)$ on a finite dimensional E-vector space V that factorizes through $\mathrm{Gal}(\overline{k}/k')$ for some finite extension k' of k. Then for every embedding $\sigma : k \to \mathbb{C}$ we have the Artin L-function $L(s, \sigma(\psi), k)$, with for every prime $\mathfrak{P}$ of k the reciprocal $\mathrm{Eul}_{\mathfrak{P}}(s, \sigma(\psi), k)$ of the Euler factor corresponding to $\mathfrak{P}$. Under the natural isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \oplus_\sigma \mathbb{C}_\sigma$ the $L(s, \sigma(\psi), k)$ correspond to a canonical $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued L-function that we denote by $L(s, \psi \otimes \mathrm{id}, k)$. Similarly, for every prime $\mathfrak{P}$ of k we have an $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued $\mathrm{Eul}_{\mathfrak{P}}(s, \psi \otimes \mathrm{id}, k)$ corresponding to the $\mathrm{Eul}_{\mathfrak{P}}(s, \sigma(\psi), k)$.

We now move on to the p-adic L-functions, and assume that k is totally real, p a prime number, a any integer and ω_p the Teichmüller character $\mathrm{Gal}(\overline{\mathbb{Q}}/k) \to \mathbb{Q}_p^*$ (2.14). If $\tau : E \to \overline{\mathbb{Q}}_p$ is an embedding and $k_{\tau(\psi)\omega_p^a}$ is totally real then from Section 2 we have the p-adic L-function $L_p(s, \tau(\psi)\omega_p^a, k)$, which is not identically zero. In this case, using the natural isomorphism $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \oplus_\tau \overline{\mathbb{Q}}_{p,\tau}$, they give us an $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$-valued p-adic L-function on $\mathbb{Z}_p$ or $\mathbb{Z}_p \setminus \{1\}$ that we denote by $L_p(s, \psi \otimes \omega_p^a, k)$.

Lemma 3.15. *The values of $L_p(s, \psi \otimes \omega_p^a, k)$ are in $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$.*

Proof. Using Brauer induction for ψ (cf. (2.10)) it suffices to do prove this when ψ is 1-dimensional. But then for each $\tau : E \to \overline{\mathbb{Q}}_p$ the function $L_p(s, \tau(\psi)\omega_p^a, k)$ can be described as in (6.3) with $l = -1$, from which the result is clear. □

Remark 3.16. It follows from Remark 2.13 that $L_p(s, \psi \otimes \omega_p^a, k)$ satisfies

$$L_p(m, \psi \otimes \omega_p^a, k) = \mathrm{Eul}_p(m, \psi \otimes \mathrm{id}, k) L(m, \psi \otimes \mathrm{id}, k)$$

for integers $m \leq 0$ congruent to $1 - a$ modulo $\phi(q)$, where both sides lie in $E = E \otimes_{\mathbb{Q}} \mathbb{Q}$ inside $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $E \otimes_{\mathbb{Q}} \mathbb{C}$ respectively, and

$$\mathrm{Eul}_p(s, \psi \otimes \mathrm{id}, k) = \prod_{\mathfrak{P}|p} \mathrm{Eul}_{\mathfrak{P}}(s, \psi \otimes \mathrm{id}, k)$$

the product being over all primes of k dividing p.

Remark 3.17. If $E = \mathbb{Q}$ we shall identify $E \otimes_{\mathbb{Q}} \mathbb{C}$ with $\mathbb{C}$, and write $L(s, \psi, k)$, etc., instead of $L(s, \psi \otimes \mathrm{id}, k)$, etc. Similarly we identify $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with $\mathbb{Q}_p$ and write $L_p(s, \psi\omega_p^a, k)$ instead of $L_p(s, \psi \otimes \omega_p^a, k)$.

We now have all the ingredients for the generalization and refinement of Conjecture 2.17. Starting with a finite Galois extension $k/\mathbb{Q}$ with Galois group G, E a finite extension of $\mathbb{Q}$, and π an idempotent in $E[G]$, we let ψ_π be the natural representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E[G]\pi$ and χ_π its associated character. If $\dim_E(E[G]\pi) = \dim_E(\pi K_{2n-1}(k)_E)$ for some $n \geq 2$ then, for any prime p and any embedding $\tau : E \to \overline{\mathbb{Q}}_p$, $\tau(\psi_\pi)\omega_p^{1-n}$ is trivial on the conjugacy class of complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by Proposition 3.8. In particular, $\mathbb{Q}_{\tau(\psi_\pi)\omega_p^{1-n}}$ is totally real and therefore $L_p(n, \chi_\pi \otimes \mathrm{id}, \mathbb{Q})$ is not identically zero. With $F \subset \overline{\mathbb{Q}}_p$ the topological closure of $\phi_p(k)$ as before, using ordered bases for $E[G]\pi$, M_π^E and $K_{2n-1}(M_\pi^E)$, we have $D(M_\pi^E)^{1/2,\infty}$ and the regulator $R_{n,\infty}(M_\pi^E)$ in $E \otimes \mathbb{C}$ as well as $D(M_\pi^E)^{1/2,p}$ and $R_{n,p}(M_\pi^E)$ in $E \otimes F$.

Conjecture 3.18. *With notation as above, if* $\dim_E(E[G]\pi) = \dim_E(\pi K_{2n-1}(k)_E)$ *for some* $n \geq 2$*, then*

(1) *in* $E \otimes_{\mathbb{Q}} \mathbb{C}$ *we have*

$$L(n, \chi_\pi \otimes \mathrm{id}, \mathbb{Q}) D(M_\pi^E)^{1/2,\infty} = e(n, M_\pi^E) R_{n,\infty}(M_\pi^E)$$

for some $e(n, M_\pi^E)$ *in* $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$*;*

(2) *in* $E \otimes_{\mathbb{Q}} F$ *we have*

$$L_p(n, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q}) D(M_\pi^E)^{1/2,p} = e_p(n, M_\pi^E) \operatorname{Eul}_p(n, \chi_\pi \otimes \mathrm{id}, \mathbb{Q}) R_{n,p}(M_\pi^E)$$

for some $e_p(n, M_\pi^E)$ *in* $(E \otimes_{\mathbb{Q}} \mathbb{Q})^*$*;*

(3) *in fact,* $e_p(n, M_\pi^E) = e(n, M_\pi^E)$*;*

(4) $L_p(n, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q})$ *and* $R_{n,p}(M_\pi^E)$ *are units in* $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ *and* $E \otimes_{\mathbb{Q}} F$ *respectively.*

Remark 3.19. (1) One sees as in the proof of Lemma 3.7 that the validity of each part of the conjecture is independent of the chosen bases of $E[G]\pi$, M_π^E and $K_{2n-1}(M_\pi^E)$.

(2) Since $E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \oplus_\sigma E_\sigma$, where the sum is over all embeddings $\sigma : E \to \mathbb{C}$, an identity in $E \otimes_{\mathbb{Q}} \mathbb{C}$ is equivalent to the corresponding identities for all such embeddings. The same holds if we replace $\mathbb{C}$ with $\overline{\mathbb{Q}}_p$.

In Remark 3.24 we shall make explicit how part (1) of this conjecture is equivalent with Beilinson's conjecture [2, Conjecture 3.4] for an Artin motive associated with π. First, we make various remarks about its dependence on E, etc., and on its relation with Conjecture 2.17.

Remark 3.20. (1) An equivalent for Conjecture 3.18(1) can be formulated for any idempotent π if $k/\mathbb{Q}$ is any finite Galois extension (see the end of Section 5), in which case it is a conjecture by Gross (see [39, p. 210]). In that case it was proved by Beilinson if the action of $G = \mathrm{Gal}(k/\mathbb{Q})$ on $E[G]\pi$ is Abelian (see loc. cit.). One can deduce that it holds for any π in $E[G]$ if this action factors through S_3 or D_8 (see Proposition 5.9).

(2) If the action of G on $E[G]$ is Abelian then parts (1)-(3) of Conjecture 3.18 also hold for any π to which the conjecture applies (see Remark 4.18).

(3) Extending and simplifying earlier work by F. Calegari [13], F. Beukers in [8] proved that $\zeta_p(2)$ is irrational when $p = 2$ or 3. This value equals $L_p(2, \chi\omega_p^{-1}, \mathbb{Q})$ with χ the primitive character on $(\mathbb{Z}/4\mathbb{Z})^*$ for $p = 2$, and the primitive character on $(\mathbb{Z}/3\mathbb{Z})^*$ for $p = 3$. Moreover, if χ is the odd primitive character on $(\mathbb{Z}/8\mathbb{Z})^*$ then he also shows that $L_2(2, \chi\omega_2^{-1}, \mathbb{Q})$ is irrational. It follows that the conjecture holds in full for π corresponding to the non-trivial representation of $\mathrm{Gal}(k/\mathbb{Q})$ and $n = 2$ when (k, p) is one of $(\mathbb{Q}(\sqrt{-1}), 2)$, $(\mathbb{Q}(\sqrt{-2}), 2)$ and $(\mathbb{Q}(\sqrt{-3}), 3)$.

(4) We have verified numerically that part (4) of the conjecture holds in certain cases; see Remark 4.19 as well as Section 7.

Remark 3.21. (1) If $\pi = \pi_1 + \cdots + \pi_m$ with $\pi_i^2 = \pi_i$ and $\pi_i\pi_j = 0$ when $i \neq j$, then the conjecture for π is implied by the conjecture for all π_i because $D(\oplus_i M_i^E)^{1,2/\infty} = \prod_i D(M_i^E)^{1/2,\infty}$, etc., as one easily sees by using bases.

(2) If π is in $E[G]$ and E' is an extension of E, then we may view π as an element of $E'[G]$ as well, and the conjectures for M_i^E and $M_i^{E'}$ are equivalent: we can use the same bases over E' as over E, so that $D(M_i^E)^{1/2,\infty} = D(M_i^{E'})^{1/2,\infty}$

in $E \otimes \mathbb{C} \subseteq E' \otimes \mathbb{C}$, and the same holds for all the other ingredients (including the L-functions).

(3) By comparing bases one sees immediately that if π and π' are conjugate under the action of $E[G]^*$ then the conjectures for π and for π' are equivalent.

(4) If, for a primitive central idempotent π_i of $E[G]$, $E[G]\pi_i \cong M_m(E)$ for some m, and π is a primitive idempotent corresponding to π_i, then the conjecture for π_i is implied by the conjecture for π. Indeed, we can decompose π_i into a sum of orthogonal primitive idempotents as in Remark 3.14(1), and the truth of the conjecture for π_i is implied by its truth for each such primitive idempotent. But all such primitive idempotents are conjugate to π hence part (3) above applies.

Remark 3.22. If $k/\mathbb{Q}$ is a Galois extension with Galois group G and H a subgroup of G with fixed field k^H, then $K_m(k)_{\mathbb{Q}}^H = K_m(k^H)_{\mathbb{Q}}$, a result known as Galois descent. With $\pi_H = |H|^{-1}\sum_{h\in H} h$, an idempotent in $E[G]$, this implies that $\pi_H K_m(k) = K_m(k^H)$.

Remark 3.23. (1) Let $k/\mathbb{Q}$ be a Galois extension with Galois group G. If N is a normal subgroup of G corresponding to $k' = k^N$ and $\pi_N = |N|^{-1}\sum_{h\in N} h$, then $\pi_N^2 = \pi_N$ and the natural map $\phi : E[G] \to E[G/N]$ has kernel $E[G](1-\pi_N)$ and induces an isomorphism $E[G]\pi_N \to E[G/N]$. Indeed, it is clear that π_N is central in $E[G]$ since N is normal in G and that $1-\pi_N$ is in the kernel of ϕ. Also, since N acts trivially on $E[G]\pi_N$ this is an $E[G/N]$-module generated by one element, so its dimension over E cannot be bigger than $|G/N|$. Since ϕ is obviously surjective our claims follow. Therefore in this situation for any idempotent π in $E[G/N]$ there is a canonical idempotent $\tilde{\pi}$ lifting π to $E[G]\pi_N$, and the natural map $E[G]\tilde{\pi} \to E[G/N]\pi$ is an isomorphism of $E[G]$- and $E[G/N]$-modules.

Then the statements of Conjecture 3.18 for π in $E[G/N]$ or for $\tilde{\pi}$ in $E[G]$ are equivalent. Namely,

$$\tilde{\pi}K_{2n-1}(k)_E = \tilde{\pi}\pi_N K_{2n-1}(k)_E = \tilde{\pi}K_{2n-1}(k')_E = \pi K_{2n-1}(k')_E$$

inside $K_{2n-1}(k)_E$ so that we can use the same bases for either side. The same holds for $\tilde{\pi}(E\otimes k)$ and $\pi(E\otimes k')$ inside $E\otimes k$. Moreover, $E[G]\tilde{\pi}$ is the pullback to G of the G/N-representation $E[G/N]\pi$, so that $L(s, E[G]\tilde{\pi}, \mathbb{Q}) = L(s, E[G/N]\pi, \mathbb{Q})$ and similarly for the p-adic L-functions.

(2) If k is a totally real number field k let $\tilde{k}$ be its (totally real) Galois closure over $\mathbb{Q}$. Then $k = \tilde{k}^H$ for some subgroup H of $G = \mathrm{Gal}(\tilde{k}/\mathbb{Q})$, $\pi = |H|^{-1}\sum_{h\in H} h$ is an idempotent in $\mathbb{Q}[G]$, and for $n \geq 2$ odd Conjecture 3.18 for π is equivalent to Conjecture 2.17 for k. Namely, $K_{2n-1}(k)_{\mathbb{Q}} = \pi K_{2n-1}(\tilde{k})_{\mathbb{Q}} \subseteq K_{2n-1}(\tilde{k})_{\mathbb{Q}}$ by Remark 3.22 and $\pi\tilde{k} = k$ so that we can use the same $\mathbb{Q}$-bases in each case. Moreover, $\mathbb{Q}[G]\pi \cong \mathrm{Ind}_H^G(1_H) = \mathbb{Q}[G]\otimes_{\mathbb{Q}[H]} 1_H$ with 1_H the trivial 1-dimensional representation of H, as one easily sees by mapping β in $\mathbb{Q}[G]\pi$ to $\beta\otimes v$ and $\sum_\sigma a_\sigma\sigma\otimes(\lambda v)$ to $\lambda\sum_\sigma a_\sigma\sigma\pi$, $\{v\}$ being a basis of 1_H. By well-known properties of Artin L-functions [40, Prop. 10.4(iv)] this implies that $\zeta_k(s) = L(n, \mathbb{Q}[G]\pi, \mathbb{Q})$ and similarly for the p-adic L-functions.

Remark 3.24. We make the relation between Conjecture 3.18(1) and Beilinson's conjecture [2, Conjecture 3.4] for (Artin) motives explicit since the relation between the two elements of E^* involved also shows up very explicitly in our computer calculations, suggesting that the element for the formulation at $s = 1-n$ is simpler

than for our formulation at $s = n$ (see Remark 7.8). We provide some details since we could not find a detailed enough reference in the literature.

With notation as in Conjecture 3.18 and (2.6) we have

$$\begin{aligned} L(1-s, \chi_\pi^\vee \otimes \mathrm{id}, \mathbb{Q}) = W(\chi_\pi^\vee \otimes \mathrm{id}) C(\chi_\pi^\vee \otimes \mathrm{id})^{s-\frac{1}{2}} (2(2\pi)^{-s}\Gamma(s))^{\chi^\vee(1)} \\ \times (\cos(\pi s/2))^{n^+} (\sin(\pi s/2))^{n^-} L(s, \chi_\pi \otimes \mathrm{id}, \mathbb{Q}), \end{aligned} \tag{3.25}$$

with $\chi_\pi^\vee$ the dual character of χ_π, $W(\chi_\pi^\vee \otimes \mathrm{id})$ in $E \otimes \mathbb{C}$, and $C(\chi_\pi^\vee \otimes \mathrm{id}) = C(\chi_\pi \otimes \mathrm{id})$ in $\mathbb{Q}^*$. If $m = \dim_E(E[G]\pi)$, then $n^+ = m$ and $n^- = 0$ for n odd, and $n^- = m$ and $n^+ = 0$ for n even. In either case $L(s, \chi_\pi^\vee \otimes \mathrm{id}, \mathbb{Q})$ has a zero of order m at $s = 1 - n$. Moreover

$$W(\chi_\pi^\vee \otimes \mathrm{id}) C(\chi_\pi^\vee \otimes \mathrm{id})^{1/2} = e_n i^{m(n-1)} D(M_\pi^E)^{1/2,\infty} \tag{3.26}$$

for some e_n in $E^* = (E \otimes_{\mathbb{Q}} \mathbb{Q})^*$ by [20, Propositions 5.5 and 6.5] since $D^{1/2,\infty}$ can be taken to be the same for M_π^E and the associated determinant representation (cf. [31, p.360]). Hence the first non-vanishing coefficient in its Taylor expansion around $s = 1 - n$, $L^\sharp(1-n, \chi_\pi^\vee \otimes \mathrm{id}, \mathbb{Q})$, equals

$$\delta_n e_n (2\pi i)^{m(1-n)} ((n-1)!/2)^m C(\chi_\pi^\vee \otimes \mathrm{id})^{n-1} D(M_\pi^E)^{1/2,\infty} L(n, \chi_\pi \otimes \mathrm{id}, \mathbb{Q})$$

with $\delta_n = (-1)^{m(n-1)/2}$ when n is odd, and $\delta_n = (-1)^{m(n+2)/2}$ when n is even. In particular, Conjecture 3.18(1) is equivalent with

$$L^\sharp(1-n, \chi_\pi^\vee \otimes \mathrm{id}, \mathbb{Q}) = \delta_n e_n ((n-1)!/2)^m C(\chi_\pi^\vee \otimes \mathrm{id})^{n-1} e(n, M_\pi^E) \widetilde{R}_{n,\infty}(M_\pi^E) \tag{3.27}$$

with the renormalized regulator $\widetilde{R}_{n,\infty}(M_\pi^E) = (2\pi i)^{m(1-n)} R_{n,\infty}(M_\pi^E)$.

Let us compare this with Beilinson's conjecture for a motive associated with π. We associate motives covariantly to smooth projective varieties over $\mathbb{Q}$ as in [2, § 2.4]. The Galois group G acts on the left on k, hence on the right on $\mathrm{Spec}(k)$, and we let $\mathcal{M}_\pi$ be the motive corresponding to π under this action. Then G acts on the left on the cohomology theories on $\mathrm{Spec}(k)$ as well as its K-theory, and after tensoring with E the corresponding groups for $\mathcal{M}_\pi$ are the images under π. Thus, the relevant motivic cohomology of $\mathcal{M}_\pi$ is $H^1_M(\mathcal{M}_{\pi/\mathbb{Z}}, \mathbb{Q}(n)) = K_{2n-1}(M_\pi^E)$.

We shall need the non-degenerate E-bilinear pairing

$$\begin{aligned} E[G] \times \bigoplus_{\tau : k \to \mathbb{C}} E &\to E \\ (\textstyle\sum_\sigma a_\sigma \sigma, (b_\tau)_\tau) &\mapsto \textstyle\sum_\sigma a_\sigma b_{\phi_\infty \circ \sigma^{-1}} . \end{aligned}$$

It factors through the left $E[G]$-action on $\oplus_\tau E$ (given by $\sigma((b_\tau)_\tau) = (b_{\tau \circ \sigma^{-1}})_\tau$), hence is trivial on $E[G](1-\pi) \times \pi(\oplus_\tau E)$ and $E[G]\pi \times (1-\pi)(\oplus_\tau E)$. We therefore obtain a non-degenerate E-bilinear pairing

$$\langle \cdot, \cdot \rangle : E[G]\pi \times \pi(\oplus_\tau E) \to E \tag{3.28}$$

that identifies $E[G]\pi$ and $\pi(\oplus_\tau E)$ as dual $E[G]$-modules.

Tensoring (2.1) with E and applying π we get

$$H^1_{\mathcal{D}}(\mathcal{M}_{\pi/\mathbb{R}}, \mathbb{R}(n)) \cong \pi\Big(\bigoplus_{\tau : k \to \mathbb{C}} E \otimes \mathbb{R}(n-1) \Big) .$$

Note that the left hand side is a subspace of the right hand side by (2.1), but because the regulator map (2.3) tensored with $\mathbb{R}$ is injective, and $\dim_E(K_{2n-1}(M_\pi^E)) = \dim_E(E[G]\pi) = \dim_E(\pi(\oplus_\tau E))$ by our assumption on π and since $D(M_\pi^E)^{1/2,\infty} \neq 0$

(as was noticed right after Definition 3.5), equality must hold. For the Beilinson regulator we therefore have to compare the two E-structures on $\det H^1_{\mathcal{D}}(\mathcal{M}_{\pi/\mathbb{R}}, \mathbb{R}(n))$ coming from Betti cohomology,

$$H^0_B(\mathcal{M}_{\pi/\mathbb{R}}, \mathbb{Q}(n-1)) \cong \pi\Big(\bigoplus_{\tau:k\to\mathbb{C}} E \otimes \mathbb{Q}(n-1)\Big) \subseteq \pi\Big(\bigoplus_{\tau:k\to\mathbb{C}} E \otimes \mathbb{R}(n-1)\Big)\,,$$

and that induced by the Beilinson regulator map (2.3),

$$H^1_M(\mathcal{M}_{\pi/\mathbb{R}}, \mathbb{Q}(n)) \to H^1_{\mathcal{D}}(\mathcal{M}_{\pi/\mathbb{R}}, \mathbb{R}(n)) = \pi\Big(\bigoplus_{\tau:k\to\mathbb{C}} E \otimes \mathbb{R}(n-1)\Big)\,.$$

Choosing an E-basis of $\pi(\oplus_\tau E)$, and multiplying it by $(2\pi i)^{n-1}$ to obtain an E-basis of $\pi(\oplus_\tau E \otimes \mathbb{Q}(n-1))$, it is easy to see that Beilinson's regulator R_{Bei} for $\mathcal{M}_\pi$ satisfies

$$R_{\mathrm{Bei}} = (2\pi i)^{m(1-n)} \det[\,\cdot\,,\,\cdot\,]_\infty / \det\langle\,\cdot\,,\,\cdot\,\rangle = \widetilde{R}_{n,\infty}(M^E_\pi) / \det\langle\,\cdot\,,\,\cdot\,\rangle$$

where all determinants are computed using the chosen E-bases.

Finally, we compare L-functions. We have

$$H^0_{\mathrm{et}}(\mathrm{Spec}(k)\otimes\overline{\mathbb{Q}}, \mathbb{Q}_l) \cong \bigoplus_{\tau:k\to\overline{\mathbb{Q}}} \mathbb{Q}_l \cong \mathbb{Q}_l \otimes_{\mathbb{Q}} \Big(\bigoplus_{\tau:k\to\overline{\mathbb{Q}}} \mathbb{Q}\Big)$$

so that $H^0_{\mathrm{et}}(\mathcal{M}_\pi, \mathbb{Q}_l) \cong \mathbb{Q}_l \otimes \pi(\oplus_\tau E)$ as $\mathbb{Q}_l \otimes E[G]$-modules. The $\mathbb{Q}_l$ plays no role, and as in (3.28) we see that $\pi(\oplus_{\tau:k\to\overline{\mathbb{Q}}} E)$ is dual to $E[G]\pi$ as $E[G]$-module. As the motivic L-function uses the geometric rather than the arithmetic Frobenius (cf. [34, p.26]), we obtain that $L(s, \mathcal{M}_\pi)$ of [2, § 3] is equal to $L(s, \chi_\pi \otimes \mathrm{id}, \mathbb{Q})$. If $\theta : E[G] \to E[G]$ is the E-linear involution obtained by replacing each σ in G with σ^{-1}, then we need to consider $\mathcal{M}^0_\pi = \mathcal{M}_{\theta(\pi)}$ instead of $\mathcal{M}_\pi$. But the map $E[G]\pi \times E[G]\theta(\pi) \to E$ mapping (α, β) to the coefficient of the neutral element of G in $\theta(\beta)\alpha$ is easily seen to identify $E[G]\theta(\pi)$ and $E[G]\pi$ as dual $E[G]$-modules, so that $L(s, \mathcal{M}^0_\pi) = L(s, \chi^\vee_\pi \otimes \mathrm{id}, \mathbb{Q})$.

4. More explicit K-groups and regulators

In this section we first describe an inductive procedure that conjecturally gives $K_{2n-1}(k)_{\mathbb{Q}}$ $(n \geq 2)$ for any number field k. It is originally due to Zagier [54], but we essentially give a reformulation by Deligne [21]. We also describe results concerning Conjecture 3.18 when the action of G on $E[G]\pi$ is Abelian.

In order to describe Zagier's conjecture we need the functions

$$\mathrm{Li}_n(z) = \sum_{k\geq 1} \frac{z^k}{k^n} \qquad (n \geq 0) \tag{4.1}$$

for z in $\mathbb{C}$ with $|z| < 1$ if $n = 0$ or 1, and $|z| \leq 1$ if $n \geq 2$. In particular, $\mathrm{Li}_1(z)$ is the main branch of $-\log(1-z)$. Using that $\mathrm{d}\,\mathrm{Li}_{n+1}(z) = \mathrm{Li}_n(z)\,\mathrm{d}\log(z)$ they extend to multi-valued analytic functions on $\mathbb{C}\setminus\{0,1\}$. By simultaneously continuing all Li_n along the same path one obtains single-valued functions on $\mathbb{C}\setminus\{0,1\}$ (see [54, § 7] or [19, Remark 5.2]) by putting

$$P_n(z) = \pi_{n-1}\Big(\sum_{j=0}^{n-1} \frac{b_j}{j!}(2\log|z|)^j \mathrm{Li}_{n-j}(z)\Big) \qquad (n \geq 1)\,, \tag{4.2}$$

with b_j the j-th Bernoulli number and π_{n-1} the projection of $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n)$ onto $\mathbb{R}(n-1)$. These functions satisfy $P_n(\bar{z}) = \overline{P_n(z)}$ as well as

$$P_n(z) + (-1)^n P_n(1/z) = 0 \tag{4.3}$$

and

$$P_n(z^m) = m^{n-1} \sum_{\zeta^m=1} P_n(\zeta z) \tag{4.4}$$

when $m \geq 1$ and $z^m \neq 1$.

We can now describe the conjecture. Let $d = [k : \mathbb{Q}]$, and for $n \geq 2$ let $\widetilde{B}_n(k)$ be a free Abelian group on generators $[x]_n^{\sim}$ with $x \neq 0, 1$ in k. Define

$$\begin{aligned} \widetilde{P}_n : \widetilde{B}_n(k) &\to \left(\mathbb{R}(n-1)^d\right)^+ \\ [x]_n^{\sim} &\mapsto (P_n(\sigma(x)))_{\sigma:k\to\mathbb{C}}\,, \end{aligned}$$

with $\left(\mathbb{R}(n-1)^d\right)^+ = \{(a_\sigma)_\sigma \text{ in } \mathbb{R}(n-1)^d \text{ such that } a_{\bar{\sigma}} = \overline{a_\sigma}\}$. Then we define inductively, for $n \geq 2$,

$$\mathrm{d}_n : \widetilde{B}_n(k) \to \begin{cases} \bigwedge^2_{\mathbb{Z}} k^* & \text{if } n = 2 \\ B_{n-1}(k) \otimes_{\mathbb{Z}} k^* & \text{if } n > 2 \end{cases}$$

by

$$[x]_n^{\sim} \mapsto \begin{cases} (1-x) \wedge x & \text{if } n = 2 \\ [x]_{n-1} \otimes x & \text{if } n > 2\,, \end{cases}$$

where $[x]_{n-1}$ denotes the class of $[x]_{n-1}^{\sim}$ in $B_{n-1}(k)$, which is defined as

$$B_n(k) = \widetilde{B}_n(k) / \operatorname{Ker}(\mathrm{d}_n) \cap \operatorname{Ker}(\widetilde{P}_n)\,.$$

There are some universal relations, one of which is that $[x]_n + (-1)^n [1/x]_n = 0$, a consequence of (4.3).

Conjecture 4.5. *If $n \geq 2$ then*

(1) *there is an injection*

$$\frac{\operatorname{Ker}(\mathrm{d}_n)}{\operatorname{Ker}(\mathrm{d}_n) \cap \operatorname{Ker}(\widetilde{P}_n)} \to K_{2n-1}(k)_{\mathbb{Q}}$$

with image a finitely generated group of rank equal to $\dim_{\mathbb{Q}}(K_{2n-1}(k)_{\mathbb{Q}})$;

(2) *Beilinson's regulator map is given by $\widetilde{P}_n$:*

$$\begin{array}{ccc} \dfrac{\operatorname{Ker}(\mathrm{d}_n)}{\operatorname{Ker}(\mathrm{d}_n) \cap \operatorname{Ker}(\widetilde{P}_n)} & \longrightarrow & K_{2n-1}(k)_{\mathbb{Q}} \\ & \searrow^{(n-1)!\widetilde{P}_n} & \downarrow \Pi_{\sigma:k\to\mathbb{C}} \operatorname{reg}_\infty \circ \sigma_* \\ & & \left(\mathbb{R}(n-1)^d\right)^+ \end{array}$$

commutes.

Remark 4.6. For $n = 2$ the corresponding results were already known before Zagier made his conjecture ([9, 51]; see also [26, § 2]).

From the results in [19, § 5] one obtains the following.

Theorem 4.7. *Let k be a number field and let $n \geq 2$ be an integer. Then there is an injection*

$$\Psi_n : \frac{\mathrm{Ker}(\mathrm{d}_n)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\widetilde{P}_n)} \to K_{2n-1}(k)_{\mathbb{Q}}$$

with finitely generated image, such that the diagram in Conjecture 4.5(2) commutes.

Remark 4.8. The part of Zagier's conjecture that remains open is the question if the rank of the image of Ψ_n equals $\dim_{\mathbb{Q}}(K_{2n-1}(k)_{\mathbb{Q}})$. For a cyclotomic field k this holds for any $n \geq 2$ as we shall recall in Example 4.10 below, but it is also known for arbitrary number fields for $n = 2$, as mentioned above, or $n = 3$ (see [25, § 3] and [27, Appendix]).

Remark 4.9. Although not stated explicitly in [19], it is clear from the construction there that the map Ψ_n in Theorem 4.7 is natural in k. In particular, if $k/\mathbb{Q}$ is Galois then $\mathrm{Gal}(k/\mathbb{Q})$ acts on $\mathrm{Ker}(\mathrm{d}_n)/\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\widetilde{P}_n)$ (through the obvious action on the generators $[x]_n^{\sim}$) as well as on $K_{2n-1}(k)_{\mathbb{Q}}$, and Ψ_n is $\mathrm{Gal}(k/\mathbb{Q})$-equivariant.

Example 4.10. If k is a cyclotomic field then Zagier's conjecture is known in full. By Theorem 4.7 it suffices to see that the rank of the image of Ψ_n equals $\dim_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}}$. For an N-th root of unity $\zeta \neq 1$, $N[\zeta]_n$ is in $\mathrm{Ker}(\mathrm{d}_n)/\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\widetilde{P}_n)$. If $k = \mathbb{Q}(\zeta)$ for a primitive N-th root of unity ζ with $N > 2$ then $\dim_{\mathbb{Q}} K_{2n-1}(k)_{\mathbb{Q}} = [k : \mathbb{Q}]/2$ for $n \geq 2$, and the $N[\zeta^j]_n$ with $1 < j < N/2$ and $\gcd(j, N) = 1$ have $\mathbb{R}$-independent images under $\widetilde{P}_n$ [54, pp.420–422] so that they are $\mathbb{Z}$-independent in $\mathrm{Ker}(\mathrm{d}_n)/\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\widetilde{P}_n)$, and the same holds for their images under the injective map Ψ_n. For $k = \mathbb{Q}$ one can easily check directly from Theorem 4.7 and (4.2) that for odd $n \geq 2$ the element $2\Psi([-1]_n)$ is non-zero.

For the p-adic regulator we need to describe the p-adic polylogarithms introduced in [16]. We first make a choice of a branch of the p-adic logarithm. Recall that a homomorphism $\log_p : \mathcal{O}^*_{\mathbb{C}_p} \to \mathbb{C}_p$ is uniquely determined by the requirement that for $|x|_p < 1$ it is given by the usual power series for $\log(1 + x)$. To extend it to a homomorphism $\log_p : {\mathbb{C}_p}^* \to \mathbb{C}_p$ it suffices to make an arbitrary choice of $\log_p(p)$. Any such extension will be called a branch of the p-adic polylogarithm. In what follows we fix one such choice.

Coleman first produced the p-adic polylogarithm out of a more extensive theory of what is now called Coleman integration. For the p-adic polylogarithm it is, however, possible to give a more elementary and explicit theory. To do this, consider the class of functions $f : \mathbb{C}_p \setminus \{1\} \to \mathbb{C}_p$ that satisfy the following properties:

(1) for any $a \in \mathbb{C}_p$ with $|a - 1|_p = 1$ we have a power series expansion for $f(z)$ in $z - a$ that converges for $|z - a|_p < 1$;
(2) for $0 < |z - 1|_p < 1$ (resp. $|z|_p > 1$) $f(z)$ is given by a polynomial in $\log_p(z - 1)$ (resp. $\log_p(1/z)$) with coefficients that are Laurent series in $z - 1$ (resp. $1/z$), convergent for $0 < |z - 1|_p < 1$ (resp. $|z|_p > 1$).

It is easy to see that differentiation sends this class surjectively onto itself. The p-adic polylogarithms are functions $\mathrm{Li}_{n,p}(z)$ $(n \geq 0)$ in this class with the following properties:

(1) $\mathrm{Li}_{0,p}(z) = \frac{z}{1-z}$;
(2) $\mathrm{Li}_{n,p}(0) = 0$ for $n \geq 0$;
(3) $\mathrm{dLi}_{n+1,p}(z) = \mathrm{Li}_{n,p}(z)\frac{\mathrm{d}z}{z}$ for $n \geq 0$;

(4) for every $n \geq 0$ there is a power series $g_n(v)$, convergent for $|v|_p < p^{1/(p-1)}$, such that $g_n(1/(1-z)) = \mathrm{Li}_{n,p}(z) - \frac{1}{p^n}\mathrm{Li}_{n,p}(z^p)$ when $|z-1|_p > p^{-1/(p-1)}$.

Note that $1/z$ has a singularity at 0 but $\mathrm{Li}_{n,p}(z)/z$ does not, because of the assumption that $\mathrm{Li}_{n,p}(0) = 0$. In fact, it is easy to see that $\mathrm{Li}_{n,p}(z)$ is given by the p-adically convergent power series (4.1) for $|z|_p < 1$.

In [6] two of the authors of the present work described an efficient algorithm for the computation of p-adic polylogarithms up to a prescribed precision. This will be used in Section 7 to compute the p-adic regulator for $n \geq 2$ that we now proceed to describe in terms of suitable combinations of $\log_p^{n-m}(z)\mathrm{Li}_m(z)$.

By Proposition 6.4 and the correct version of Proposition 6.1 of [16] the $\mathrm{Li}_{n,p}(z)$ for $n \geq 0$ and z in $\mathbb{C}_p \setminus \{1\}$ satisfy

$$\mathrm{Li}_{n,p}(z) + (-1)^n \mathrm{Li}_{n,p}(1/z) = -\frac{1}{n!}\log_p^n(z)$$

when $z \neq 0$, and

$$\mathrm{Li}_{n,p}(z^m) = m^{n-1} \sum_{\zeta^m=1} \mathrm{Li}_{n,p}(\zeta z) \tag{4.11}$$

when $m \geq 1$ and $z^m \neq 1$.

For z in $\mathbb{C}_p \setminus \{0,1\}$ and fixed $n \geq 2$, we define

$$P_{n,p}(z) = \sum_{j=0}^{n-1} c_j \log_p^j(z)\mathrm{Li}_{n-j}(z) \qquad (n \geq 2) \tag{4.12}$$

for any c_j in $\mathbb{C}_p$ satisfying $c_0 = 1$ and $\sum_{j=0}^{n-1} c_j/(n-j)! = 0$, so that $P_{n,p}(z) + (-1)^n P_{n,p}(1/z) = 0$. This is the case for $c_j = b_j$ with b_j the j-th Bernoulli numbers as before, although the resulting formula is different from (4.2). Another possible natural candidate for the function $P_{n,p}(z)$ is $L_n(z) + L_{n-1}(z)\log_p(z)/n$, where

$$L_n(z) = \sum_{m=0}^{n-1} \frac{(-1)^m}{m!}\log_p^m(z)\mathrm{Li}_{n-m}(z)$$

(see [5, (1.3) and Remark 1.5]).

The relations corresponding to (4.3) and (4.11) for $P_{n,p}(z)$ $(n \geq 2)$ are then

$$P_{n,p}(z) + (-1)^n P_{n,p}(1/z) = 0$$

when $z \neq 0, 1$, and

$$P_{n,p}(z^m) = m^{n-1} \sum_{\zeta^m=1} P_{n,p}(\zeta z) \tag{4.13}$$

when $m \geq 1$ and $z^m \neq 0, 1$.

For any branch of the p-adic logarithm one then has the following result for the p-adic regulator map (see [5, Theorem 1.10 or page 909]). Since its formulation depends on the embedding of k we deal with one such embedding at a time.

Theorem 4.14. *Let k be a number field, and let $F \subset \overline{\mathbb{Q}}_p$ be the topological closure of the Galois closure of any embedding $k \to \overline{\mathbb{Q}}_p$. For $\sigma : k \to F$ let*

$$B_n^\sigma(k) = \langle [x]_n \mid \sigma(x),\, 1-\sigma(x) \text{ are in } \mathcal{O}_{\mathbb{C}_p}^* \rangle \subseteq B_n(k) = \frac{\widetilde{B}_n(k)}{\mathrm{Ker}(\mathrm{d}_n) \cap \mathrm{Ker}(\widetilde{P}_n)}\,.$$

Then

$$\begin{aligned}\widetilde{B}_n(k) &\to F\\ [x]_n &\mapsto P_{n,p}(\sigma(x))\end{aligned}$$

induces a map

$$P^{\sigma}_{n,p} : B^{\sigma}_n(k) \to F$$

and the solid arrows in

$$\begin{array}{ccccc} B^{\sigma}_n(k)\cap \mathrm{Ker}(\mathrm{d}_n) & \hookrightarrow & \dfrac{\mathrm{Ker}(\mathrm{d}_n)}{\mathrm{Ker}(\mathrm{d}_n)\cap \mathrm{Ker}(\widetilde{P}_n)} & \xrightarrow{\Psi_n} & K_{2n-1}(k)_{\mathbb{Q}} \\ & \searrow^{(n-1)!P^{\sigma}_{n,p}} & & \cdots\!\searrow^{(n-1)!P^{\sigma}_{n,p}} & \downarrow{\scriptstyle \mathrm{reg}_p\circ\sigma_*} \\ & & & & F \end{array}$$

form a commutative diagram.

Remark 4.15. It was conjectured in [5, Conjecture 1.14] that the dotted arrow exists and that the full diagram commutes. This is known to hold for $N[\zeta]_n$ if ζ is any N-th root of unity other than 1 (see Theorem 1.12 of loc. cit.).

Remark 4.16. The formulae for the complex and p-adic regulators involved in Theorems 4.7 and 4.14 are stated up to sign in [19, Proposition 4.1] (with a normalizing factor) and [5, Theorem 1.10(2) or Proposition 7.14] respectively, since they depend on the choice of Ψ_n, which is natural only up to sign. In the p-adic case it is clear from [5, Proposition 7.10] that the sign is $(-1)^n$ if the "relativity isomorphism" in the p-adic case is normalized as in Proposition 5.7 of loc. cit. and the K-theoretic relativity isomorphism as in (3.1) of loc. cit. corresponds to this under the naturality of the regulator map. In order to see that the sign for the complex regulator is the same with the same choice of Ψ_n, we observe that one can use the techniques of [5, Appendix A and Section 5] to describe the target for the complex regulator of $[z]_n$ with z in $\mathbb{C}\backslash\{0,1\}$ in [19] by means of a complex with the same formal structure as in [5, Definition 5.2], but using $\mathbb{R}(n-1)$-valued C^∞-forms ψ with logarithmic poles on copies of $(\mathbb{P}^1_{\mathbb{C}}\backslash\{1,z\})^j$ indexed by increasing functions $f : [1,\dots,j] \to [1,\dots,n-1]$. Using the maps $\psi \mapsto \int_{(\mathbb{P}^1_{\mathbb{C}})^j} \mathrm{d}\arg(t_{f(1)}) \wedge \cdots \wedge \mathrm{d}\arg(t_{f(j)}) \wedge \psi$ for a j-form ψ with index $f : [1,\dots,j] \to [1,\dots,n-1]$, and suitable normalizing factors, one writes down an $\mathbb{R}$-linear map Π_∞ on such forms, with values in $\mathbb{R}(n-1)$, that vanishes on exact forms and satisfies the analogous normalization condition as $\Pi = \Pi_p$ in [5, Proposition 5.7]. Moreover, as in [19, Section 2.5] one sees that the subcomplex of forms on the component where $j = n-1$ that vanish if some $t_l = 0$ or ∞ gives a quasi-isomorphism under inclusion, and that the image of the regulator of $[z]_n$ is given by $\varepsilon_n(z)$ as in Section 4 of loc. cit. (up to sign since one has to use the correct formula for ω_n, which can be obtained as in [5, Section 6]). Using the computations on pages 236 and 237] of [19] one then sees that Π_∞ maps $\varepsilon_n(z)$ to $(-1)^n(n-1)!P^\flat_n(z)$, where $P^\flat_n(z) = \pi_{n-1}\left(\sum_{l=0}^{n-1} \frac{1}{l!}(-\log|z|)^l \mathrm{Li}_{n-l}(z)\right)$, which induces the same map on $\mathrm{Ker}(\mathrm{d}_n)/\mathrm{Ker}(\mathrm{d}_n)\cap\mathrm{Ker}(\widetilde{P}_n)$ as $(-1)^n(n-1)!\,P_n(z)$. Since the relativity isomorphism in K-theory only depends on a choice of $t_l = 0$ or ∞ for $l = 1,\dots,n-1$ this shows that the signs in the formulae for the complex and p-adic regulators of $[z]_n$ match for the same choice of Ψ_n, and we adjust Ψ_n so that those signs disappear.

We conclude this section by considering Conjecture 3.18 when the action of G on $E[G]\pi$ is Abelian.

Proposition 4.17. *Let $N \geq 2$, $k = \mathbb{Q}(\mu_N)$, $G = \mathrm{Gal}(k/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^*$ where a in $(\mathbb{Z}/N\mathbb{Z})^*$ corresponds to ψ_a in G satisfying $\psi_a(\zeta) = \zeta^a$ for all roots of unity ζ in k. Assume that E contains a root of unity of order equal to the exponent of G and let π in $E[G]$ be the idempotent corresponding to an irreducible character χ of G. Then parts (1), (2) and (3) of Conjecture 3.18 hold for π and $n \geq 2$ if $\chi(-1) = (-1)^{n-1}$.*

Proof. That Conjecture 3.18(1) holds in this case is well-known (cf. [39]) but we make it explicit here for part (3) of the conjecture (cf. [30, Proposition 3.1]).

We deal with the case $N = 2$ later so by Remark 3.23(1) we may assume $N > 2$ and χ primitive. The idempotent π corresponding to χ is $|G|^{-1} \sum_{a=1}^{\prime N} \chi^{-1}(a) \otimes \psi_a$, where the prime indicates that we sum over a that are relatively prime to N.

Note that by our assumption on $\chi(-1)$ and Proposition 3.8 the dimension of $K_{2n-1}(M_\pi^E)$ equals 1. Fixing a generator ζ of $\mu_N \subset k^*$ we use π, $\pi(1 \otimes \zeta)$ and $\pi(1 \otimes [\zeta]_n)$ as basis vectors of $E[G]\pi$, M_π^E and $K_{2n-1}(M_\pi^E)$ respectively, where we simplify notation by writing $[\zeta]_n$ instead of $\Psi_n([\zeta]_n)$ as in Theorem 4.7.

In order to verify (1) in $E \otimes_{\mathbb{Q}} \mathbb{C}$ it suffices by Remark 3.19(2) to consider all embeddings $\sigma : E \to \mathbb{C}$. If $\eta = \phi_\infty(\zeta)$ with ϕ_∞ as in Section 3 then the σ-component of $D(M_\pi^E)^{1/2,\infty}$ becomes

$$D(M_\pi^E)_\sigma^{1/2,\infty} = |G|^{-1} \sum_{a=1}^{N}{}' \chi_\sigma^{-1}(a)\eta^a = N|G|^{-1} \left(\sum_{a=1}^{N}{}' \chi_\sigma(a)\eta^{-a} \right)^{-1}$$

by [36, (3) on p.84]. Similarly, by Theorem 4.7 the σ-component of $R_{n,\infty}(M_\pi^E)$ is

$$R_{n,\infty}(M_\pi^E)_\sigma = (n-1)!\,|G|^{-1} \sum_{a=1}^{N}{}' \chi_\sigma^{-1}(a) P_n(\eta^a) = (n-1)!\,|G|^{-1} \sum_{a=1}^{N}{}' \chi_\sigma^{-1}(a) \mathrm{Li}_n(\eta^a)$$

because $P_n(\eta^a) = (\mathrm{Li}_n(\eta^a) + \chi_\sigma^{-1}(-1)\mathrm{Li}_n(\eta^{-a}))/2$ by our assumption on $\chi(-1)$, where Li_n is computed using the power series in (4.1). They relate to the σ-component of $L(n, \chi, \mathbb{Q})$ via $L(n, \chi_\sigma, \mathbb{Q}) = (n-1)!^{-1} R_{n,\infty}(M_\pi^E)_\sigma / D(M_\pi^E)_\sigma^{1/2,\infty}$ according to (2) on page 172 of [16] with the correct sign in the exponent of the Gauss sum as used here (cf. [54, p. 421]). Since $L(n, \chi_\sigma, \mathbb{Q}) \neq 0$ this also shows that $\pi(1 \otimes [\zeta]_n)$ gives a basis of $K_{2n-1}(M_\pi^E)$ as claimed.

Similarly, for (2) and (3) we consider all embeddings $\tau : E \to \overline{\mathbb{Q}}_p$. If $\phi_p(\zeta) = \eta_p$ then the τ-component of $D(M_\pi^E)^{1/2,p}$ is $N|G|^{-1}(\sum_{a=1}^{\prime N} \chi_\tau(a)\eta_p^{-a})^{-1}$ and the τ-component of $R_{n,p}(M_\pi^E)$ equals $(n-1)!\,|G|^{-1} \sum_{a=1}^{\prime N} \chi_\tau^{-1}(a) P_{n,p}(\eta_p^a)$ by Remark 4.15, independent of our assumption on $\chi(-1)$. By (3) on page 172 of [16] we have

$$L_p(n, \chi_\tau \omega_p^{1-n}, \mathbb{Q}) = (n-1)!^{-1} \mathrm{Eul}_p(k, \chi_\tau, \mathbb{Q}) R_{n,p}(M_\pi^E)_\tau / D(M_\pi^E)_\tau^{1/2,p}\,.$$

Therefore Conjecture 3.18(1)-(3) hold with $e(n, M_\pi^E) = e_p(n, M_\pi^E)$ in $\mathbb{Q}^* \subseteq E^*$.

Now assume that $N = 2$ so that χ is the trivial character 1, $\pi = 1$ and $n \geq 2$ is odd. Taking 1, $1 \otimes 1$ and $1 \otimes [-1]_n$ as bases of $E[G] = E$, $M_\pi^E = E \otimes \mathbb{Q}$ and $K_{2n-1}(\mathbb{Q})_E$ we find from Theorem 4.7, (4.1), (4.2) and (4.4) that

$$L(n, 1, \mathbb{Q}) = \zeta(n) = \frac{2^{n-1}}{1 - 2^{n-1}} P_n(-1) = \frac{2^{n-1}}{(n-1)!(1 - 2^{n-1})} R_{n,\infty}(M_1^E)\,,$$

again proving that $[-1]_n$ gives a basis for $K_{2n-1}(\mathbb{Q})_{\mathbb{Q}}$. Coleman proves that

$$L_p(n,\omega_p^{1-n},\mathbb{Q}) = (1-p^{-n})\lim_{x\to 1}{}' \operatorname{Li}_{n,p}(x)$$

where the limit is taken in any subfield of $\mathbb{C}_p$ that is of finite ramification degree over $\mathbb{Q}_p$. But by (4.12) and (4.13) we can rewrite this as

$$L_p(n,\omega_p^{1-n},\mathbb{Q}) = \frac{(1-p^{-n})2^{n-1}}{1-2^{n-1}}P_{n,p}(-1) = \frac{(1-p^{-n})2^{n-1}}{(n-1)!(1-2^{n-1})}R^p_{n,\infty}(M_1)\,.$$

□

Remark 4.18. The π in the proof of Proposition 4.17 are primitive in the sense of Definition 3.13. So, by Remarks 3.21(4) and 3.23(1), if Conjecture 3.18 applies to π in $E[G]$, then parts (1), (2) and (3) of it hold for π for any E if the action on $E[G]\pi$ is Abelian. In particular, parts (1) and (2) of Conjecture 2.17 hold for any totally real Abelian number field by Remark 3.21.

Remark 4.19. Computer calculations show that $L_p(n,\omega_p^{1-n},\mathbb{Q})$ is in $\mathbb{Q}_p^*$ when $p = 2,\dots,19$ and $n = 2,\dots,20$ is odd, verifying Conjecture 2.17(3) in those cases. Similarly, with notation and assumptions as in Proposition 4.17, for the 470 primitive characters χ of $\operatorname{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^*$ with $2 \le N \le 50$, $L_p(n,\chi\otimes\omega_p^{1-n},\mathbb{Q})$ lies in $(E\otimes\mathbb{Q}_p)^*$ for those values of p and $n = 2,\dots,20$ whenever $\chi(-1) = (-1)^{n-1}$. Thus Conjecture 3.18(4) also holds in those cases.

5. Computing K-groups in practice

Let $n \ge 2$. It follows from Theorem 4.7 that Zagier's approach as described in Section 4 can be used to obtain at least some part of $K_{2n-1}(k)_{\mathbb{Q}}$. Using the notation introduced in Section 4, in order to carry this approach out in practice one starts with two sets $S \subseteq S'$ of primes of $\mathcal{O}_k$ and considers only $[x]_n^\sim$ where x is an S-unit and $1-x$ is an S'-unit. When we consider several hundred of such $[x]_n^\sim$'s (and avoid using both $[x]_n^\sim$ and $[1/x]_n^\sim$ since $[x]_n + (-1)^n[1/x]_n = 0$ as mentioned in Section 4) this method is well suited to computer calculations. The only point in Zagier's approach that cannot be carried out algebraically is to determine which elements in $\operatorname{Ker}(\tilde{\mathrm{d}}_n)$ are actually in $\operatorname{Ker}(\tilde{\mathrm{d}}_n)\cap\operatorname{Ker}(\widetilde{P}_n)$. This is done using standard methods for finding linear relations between the $\widetilde{P}_n(\alpha_j)$'s for a basis $\{\alpha_j\}$ of $\operatorname{Ker}(\tilde{\mathrm{d}}_n)$ with small integral coefficients. Typically, we used about the first 50 decimals after the decimal point of all the polylogarithms of all elements of k involved in the α_j embedded into $\mathbb{C}$ in all possible ways. The relations that were found this way were then verified to hold up to at least 30 additional decimal places. As a final check, when $\operatorname{Ker}(\mathrm{d}_n)$ had the same rank as $K_{2n-1}(k)$ the number q in (2.4) (but now using the regulator $\widetilde{V}_n(k)$ for the subgroup of $K_{2n-1}(k)_{\mathbb{Q}}$ we obtained rather than $V_n(k)$; cf. Remark 2.19) was computed with a working precision of up to 120 decimals. It turned out that its reciprocal looked rational rather convincingly.

For example, for the cubic field $k = \mathbb{Q}[x]/(x^3-x^2-3x+1)$ with discriminant 148, one of the fields used in Example 7.1, we find for $\zeta_k(n)|D_k|^{1/2}/\widetilde{V}_n(k)$ the values

$$-1.75310445580715850986121256391526661796932067 22\times 10^{-2} \qquad (n=3)$$

$$-4.11706858840625185494525250643677324558351677 54\times 10^{-9} \qquad (n=5)$$

with reciprocals

$$-57.041666666666666666666666666666666666666666666662 \qquad (n=3)$$
$$-242891265.599999999999999999999999999999999999998 \qquad (n=5)\,.$$

As the calculation of a basis of $K_{2n-1}(k)_{\mathbb{Q}}$, using the method outlined above, tends to take (substantially) longer when $[k:\mathbb{Q}]$, the absolute value of the discriminant of k, or n increase, we describe two straightforward methods to reduce this. They, and most of our arguments below, will rely on Remark 3.22.

Method 5.1. *If k is an Abelian extension of $\mathbb{Q}$, hence is contained in a cyclotomic field k', then we can find $K_{2n-1}(k)_{\mathbb{Q}}$ by computing $K_{2n-1}(k')_{\mathbb{Q}}$ as in Example 4.10 and applying Remark 3.22 (using Remark 4.9).*

Since parts (1), (2) and (3) of Conjecture 3.18 are known for all idempotents of $E[G]$ to which it applies if G is Abelian by Remark 3.21 and Proposition 4.17, this will not be used for verifying Conjecture 3.18. Instead, in the cases that we shall consider we rely on the following method in order to find the K-groups.

Method 5.2. *Let $k/\mathbb{Q}$ be Galois with Galois group G, $E/\mathbb{Q}$ any extension, and M an irreducible $E[G]$-module in $K_{2n-1}(k)_E$. If H is a subgroup of G with $M^H \neq 0$ then $M \subseteq E[G]\cdot K_{2n-1}(k^H)_{\mathbb{Q}}$. If for every irreducible M we can take $H \neq \{e\}$, then we reduce to finding $K_{2n-1}(l)_{\mathbb{Q}}$ together with the action of G on it inside $K_{2n-1}(k)_{\mathbb{Q}}$ for proper subfields l of k.*

We now discuss in Examples 5.3, 5.5 and 5.7 how to use Method 5.2 for certain Galois extensions $k/\mathbb{Q}$. In the decomposition of $K_{2n-1}(k)_E$ according to central idempotents π_i of $E[G]$ as in Section 3 we concentrate on those primitive central π_i for which the action on $E[G]\pi_i$ is not Abelian and describe corresponding primitive idempotents π (as in Definition 3.13(2)) for later use. Due to restrictions on when we can calculate the p-adic L-functions numerically (see Section 6) we only consider k that are Abelian over a quadratic subfield. Of course Proposition 3.12 describes when Conjecture 3.18 applies to π, but we work out the structure of the K-groups and the Galois action in more detail.

Example 5.3. Let $k/\mathbb{Q}$ be an S_3-extension with quadratic subextension q. With $E=\mathbb{Q}$ and σ in G of order three, $\pi_1 = (e+\sigma+\sigma^2)/3$ and $\pi_2 = 1-\pi_1$ are orthogonal central idempotents with π_2 primitive (as in Definition 3.13(2)). In $\mathbb{Q}[G] = \mathbb{Q}[G]\pi_1 \oplus \mathbb{Q}[G]\pi_2$ the first summand consists of Abelian representations of G and the last of two copies of the irreducible 2-dimensional representation V of G. A corresponding primitive idempotent for π_2 is $\pi = \pi_2\pi_H$ for any subgroup $H=\langle\tau\rangle$ of G of order 2 where $\pi_H = (e+\tau)/2$.

By Remark 3.22 we have $\pi_1 K_{2n-1}(k)_{\mathbb{Q}} = K_{2n-1}(q)_{\mathbb{Q}}$ so that

$$K_{2n-1}(k)_{\mathbb{Q}} = K_{2n-1}(q)_{\mathbb{Q}} \oplus \pi_2 K_{2n-1}(k)_{\mathbb{Q}}\,. \tag{5.4}$$

The last summand is a $\mathbb{Q}[G]$-module isomorphic to V^{t_n} for some $t_n \geq 0$. As $V = \pi V \oplus \tau'\pi V$ for any $\tau' \neq \tau$ of order 2, we find, if $c = k^H$,

$$\pi_2 K_{2n-1}(k)_{\mathbb{Q}} = \pi_2 K_{2n-1}(c)_{\mathbb{Q}} \oplus \tau'\pi_2 K_{2n-1}(c)_{\mathbb{Q}}$$

again by Remark 3.22. So we reduce to the calculation of $K_{2n-1}(q)_{\mathbb{Q}}$ and $K_{2n-1}(c)_{\mathbb{Q}}$ together with the action of G on the latter.

A dimension count in (5.4) using Theorem 2.2 determines t_n. We distinguish two cases.

Case 1. If q is real then $t_n = 0$ for n even and 2 for n odd.
Case 2. If q is imaginary then $t_n = 1$.
Conjecture 3.18 applies to π when $t_n = 2$.

Example 5.5. Let $k/\mathbb{Q}$ be a D_8-extension where $D_8 = \langle \sigma, \tau | \sigma^4 = \tau^2 = (\sigma\tau)^2 = e \rangle$. We fix an isomorphism $G = \mathrm{Gal}(k/\mathbb{Q}) \cong D_8$ and use notation as in Figure 5.1.

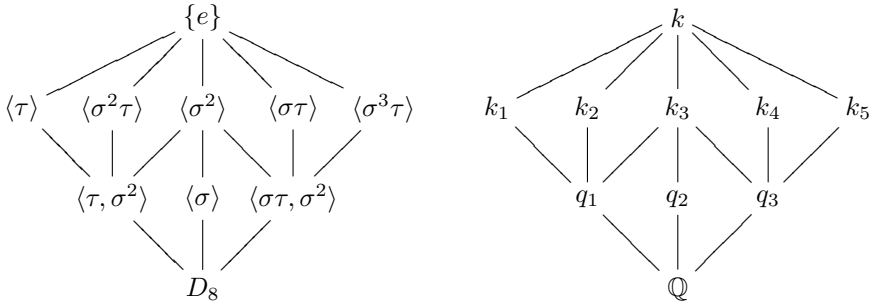


FIGURE 5.1. The subgroups of $G = D_8$ and their fixed fields in $k/\mathbb{Q}$.

Then $\mathbb{Q}[G] = \mathbb{Q}[G]\pi_1 \oplus \mathbb{Q}[G]\pi_2$ where $\pi_1 = (1+\sigma^2)/2$ and $\pi_2 = 1-\pi_1$ are orthogonal central idempotents. The first term is an Abelian representation of G and the second as a $\mathbb{Q}[G]$-module is isomorphic to V^2 for V the irreducible 2-dimensional representation of G over $\mathbb{Q}$. Then π_2 is a primitive central idempotent and $\pi = \pi_2\pi_H$ a corresponding primitive idempotent if we let $H = \langle \tau \rangle$ and $\pi_H = (1+\tau)/2$.

Since $\pi_1 K_{2n-1}(k)_{\mathbb{Q}} = K_{2n-1}(k_3)_{\mathbb{Q}}$ by Remark 3.22, for $K_{2n-1}(k)_{\mathbb{Q}}$ we find

$$K_{2n-1}(k)_{\mathbb{Q}} = K_{2n-1}(k_3)_{\mathbb{Q}} \oplus \pi_2 K_{2n-1}(k)_{\mathbb{Q}} \tag{5.6}$$

with the second term a $\mathbb{Q}[G]$-module isomorphic with V^{t_n} for some $t_n \geq 0$. If τ' in G has order two and does not commute with τ then $V = \pi V \oplus \tau'\pi V$, so by Remark 3.22 we obtain

$$K_{2n-1}(k)_{\mathbb{Q}} = K_{2n-1}(k_3)_{\mathbb{Q}} \oplus \pi_2 K_{2n-1}(k_1)_{\mathbb{Q}} \oplus \tau'\pi_2 K_{2n-1}(k_1)_{\mathbb{Q}}$$

and we reduce to the calculation of $K_{2n-1}(k_3)_{\mathbb{Q}}$ and $K_{2n-1}(k_1)_{\mathbb{Q}}$ together with the action of G on the latter.

We can find t_n by counting dimensions in (5.6), and distinguish three cases.
Case 1. For k_3 and k totally real $t_n = 0$ for n even and $t_n = 2$ for n odd.
Case 2. For k_3 totally real but k not $t_n = 2$ for n even and $t_n = 0$ for n odd.
Case 3. For k_3 not totally real $t_n = 1$.
Again Conjecture 3.18 applies to π when $t_n = 2$.

Example 5.7. Let $k/\mathbb{Q}$ be an $S_3 \times \mathbb{Z}/3\mathbb{Z}$-extension, identify $G = \mathrm{Gal}(k/\mathbb{Q})$ and $S_3 \times \mathbb{Z}/3\mathbb{Z}$ via a fixed isomorphism, and choose a generator σ of $A_3 \subset S_3$. Let q be the quadratic subfield of k, s' the fixed field of $A_3 \times \{0\}$, s'' the fixed field of $\{e\} \times \mathbb{Z}/3\mathbb{Z}$ and s_i for $i = 1, 2$ the fixed field of $H_i = \langle (\sigma, i) \rangle$.

If E is an extension of $\mathbb{Q}$ that contains a primitive third root of unity ζ_3 then

$$\begin{aligned} E[G] &= E[G]\pi_1 \oplus E[G]\pi_2 \oplus E[G]\pi_3 \oplus E[G]\pi_4 \\ &\cong E[\mathbb{Z}/6\mathbb{Z}] \oplus W_1^2 \oplus W_{\zeta_3}^2 \oplus W_{\zeta_3^2}^2 \end{aligned} \tag{5.8}$$

where

$$\pi_1 = ((e,0)+(\sigma,0)+(\sigma^2,0))/3$$
$$\pi_2 = (1-\pi_1)((e,0)+(e,1)+(e,2))/3$$
$$\pi_3 = (1-\pi_1)((e,0)+\zeta_3^2(e,1)+\zeta_3(e,2))/3$$
$$\pi_4 = (1-\pi_1)((e,0)+\zeta_3(e,1)+\zeta_3^2(e,2))/3$$

are orthogonal central idempotents of $E[G]$ satisfying $\pi_1+\pi_2+\pi_3+\pi_4=(e,0)$, and W_a is the irreducible representation of G given by $(\rho,m)(v)=a^m\rho(v)$ for v in the irreducible 2-dimensional representation of S_3. With $\pi_{H_i}=\frac{1}{3}\sum_{h\in H_i}h$, a primitive idempotent corresponding to the primitive central idempotent π_j ($j=3,\,4$) is $\pi=\pi_j\pi_{H_i}$.

Using Remark 3.22 for $A_3\times\{0\}$ and $\{e\}\times\mathbb{Z}/3\mathbb{Z}$ we see that

$$\begin{aligned}K_{2n-1}(k)_E &= \pi_1K_{2n-1}(k)_E\oplus\pi_2K_{2n-1}(k)_E\oplus\pi_3K_{2n-1}(k)_E\oplus\pi_4K_{2n-1}(k)_E\\ &= K_{2n-1}(s')_E\oplus(1-\pi_1)K_{2n-1}(s'')_E\oplus\pi_3K_{2n-1}(k)_E\oplus\pi_4K_{2n-1}(k)_E\,.\end{aligned}$$

Because $W_a=W_a^{H_1}\oplus W_a^{H_2}$ when $a\neq 1$ we have

$$\pi_jK_{2n-1}(k)_E=\pi_jK_{2n-1}(s_1)_E\oplus\pi_jK_{2n-1}(s_2)_E\qquad(j=3,4)$$

so that we reduce to the calculation of $K_{2n-1}(s)_{\mathbb{Q}}$ for the sextic subfields s of k, together with the action of G when $s=s''$, s_1 or s_2.

We have $\pi_3K_{2n-1}(k)_E\cong W_{\zeta_3}^{t_n}$ and $\pi_4K_{2n-1}(k)_E\cong W_{\zeta_3^2}^{t_n}$ for the same t_n as one sees by taking $E=\mathbb{Q}(\zeta_3)$ and considering the obvious action of $\mathrm{Gal}(E/\mathbb{Q})$ on $E[G]$. Since the dimension of $(1-\pi_1)K_{2n-1}(s'')_E$ was obtained in Example 5.3 we get the following results for t_n.

Case 1. For q real $t_n=0$ when n is even and $t_n=2$ when n is odd.

Case 2. For q imaginary $t_n=1$.

Conjecture 3.18 applies to π corresponding to π_j ($j=3,\,4$) when $t_n=2$.

We finish this section with a result that was promised in Remark 3.20(1). Note that the assumption $\dim_E(E[G]\pi)=\dim_E(\pi K_{2n-1}(k)_E)$ is not needed here: we consider the regulator map with values in $\pi(\oplus_{\tau:k\to\mathbb{C}}E\otimes\mathbb{R}(n-1))^+$ obtained from (2.3), and use Beilinson's regulator R_{Bei} comparing E-structures as in Remark 3.24. Beilinson's conjecture [2, Conjecture 3.4] states that the order of vanishing at $s=1-n$ of $L(s,\chi_\pi^\vee\otimes\mathrm{id},\mathbb{Q})$ should be $\dim_E(\pi K_{2n-1}(k)_E)$ and that the first non-vanishing coefficient at $s=1-n$ equals

$$L^\sharp(1-n,\chi_\pi^\vee\otimes\mathrm{id},\mathbb{Q})=eR_{\mathrm{Bei}}$$

for some e in E^*.

Proposition 5.9. *Let k be a $G=S_3$-extension (resp. $G=D_8$-extension) of $\mathbb{Q}$, and let $n\geq 2$. Then this conjecture holds for any primitive idempotent π corresponding to π_2 in Example 5.3 (resp. Example 5.5).*

Proof. Clearly it suffices to prove this when $E=\mathbb{Q}$. If $G=S_3$ then by Remark 3.21(3) we may assume $\pi_H=\pi_1\pi_H+\pi$ with $\pi_1\pi_H$ and $\pi=\pi_2\pi_H$ orthogonal idempotents, where we use notation as in Example 5.3. The conjecture for π_H follows from Theorem 2.2 for $c=k^H$ by the arguments of Remark 3.23 and the functional equation (2.6) of $\zeta_c(s)$. Similarly, it holds for $\pi_1\pi_H=\frac{1}{6}\sum_{g\in G}g$, where it corresponds to that theorem for $k^G=\mathbb{Q}$, and it follows that it must hold for π as well. If $G=D_8$ the argument is identical, using Theorem 2.2 for k_1 and q_1. □

6. Computing p-adic L-functions

We briefly sketch the method used to compute values of p-adic L-functions of 1-dimensional Artin characters. Values of p-adic L-functions of characters of higher dimension can then be deduced using (2.11). The method we employ is somewhat similar to, but more technical than the construction in [18], and generalizes the one used in [45] to compute values of p-adic L-functions of real quadratic fields at $s = 1$. Note that several modifications are necessary compared to [18] since the construction there is for partial ζ-functions of the ray class modulo p whereas we need to work with partial ζ-functions of arbitrary class groups. Also, we need to replace $p = 2$ in a certain number of results by $q = 4$. We leave it to the careful reader to check that these changes are indeed possible, or refer to the forthcoming article [44] where the details of the actual method used will be given.

Let k be a totally real number field and let $d = [k : \mathbb{Q}]$. Let χ be a $\overline{\mathbb{Q}}_p$-valued 1-dimensional Artin character of $\mathrm{Gal}(\overline{k}/k)$. As in Section 2, we let $q = p$ if p is odd, and $q = 4$ if $p = 2$.

By class field theory, χ can be viewed as a character of a suitable ray class group of k. We denote by $\mathfrak{f}$ the conductor of χ, which is an integral ideal of k, and we let $\mathfrak{g}$ be the modulus with infinite part $\mathfrak{g}_\infty$ consisting of all infinite places of k, and finite part $\mathfrak{g}_0$ equal to the least common multiple of $\mathfrak{f}$ and q. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ be representatives of the classes of the ray class group of k modulo $\mathfrak{g}$, and let $\zeta_{\mathfrak{a}_i}(s)$, $i = 1, \dots, h$, be the corresponding partial ζ-functions. According to [40, Chapter VII, p.526] we have, for any isomorphism $\sigma : \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\sum_{i=1}^{h} \sigma \circ \chi(\mathfrak{a}_i)\zeta_{\mathfrak{a}_i}(s) = \mathrm{Eul}_p(s, \sigma \circ \chi, k)L(s, \sigma \circ \chi, k)\,.$$

Since all $\zeta_{\mathfrak{a}_i}(m)$ are rational for integers $m \leq 0$ by loc. cit. Chapter VII, Corollary 9.9, we may identify them with $\sigma^{-1}(\zeta_{\mathfrak{a}_i}(m))$ in $\mathbb{Q}_p$. Thus for such m we have

$$\sum_{i=1}^{h} \chi(\mathfrak{a}_i)\zeta_{\mathfrak{a}_i}(m) = \mathrm{Eul}_p^*(m, \chi, k)L^*(m, \chi, k) \tag{6.1}$$

where $L^*(m, \chi, k)$ and $\mathrm{Eul}_p^*(m, \chi, k)$ are as in (2.7) and (2.8).

Now let $\beta \neq 1$ in $\mathcal{O}_k$ be such that

(1) $\beta \equiv 1 \pmod{\mathfrak{g}_0}$ and $v(\beta) > 0$ for every infinite place v of k;
(2) $\mathcal{O}_k/(\beta) \simeq \mathbb{Z}/b\mathbb{Z}$, where $b = \mathcal{N}(\beta)$ is the norm of the principal ideal (β).

Scaling the measures obtained in [18, Lemme 4.4] for the $\mathfrak{a}_i$ we obtain $\mathbb{Q}_p$-valued measures $\widetilde{\lambda}_i$ ($i = 1, \dots, d$) on $\mathbb{Z}_p^d$, depending on β and $\mathfrak{a}_i$, such that for all integers $m \leq 0$,

$$(b^{m-1} - 1)\zeta_{\mathfrak{a}_i}(m) = (-1)^{dm}\mathcal{N}(\mathfrak{a}_i)^{-m} \int (x_1 \cdots x_d)^{-m}\, d\widetilde{\lambda}_i\,.$$

The proof of Lemma 4.3 of loc. cit. shows, contrary to the statement of that lemma, that the measures $\widetilde{\lambda}_i$ are supported on $(-1 + q\mathbb{Z}_p)^d$. Therefore pulling back over multiplication by -1 on $\mathbb{Z}_p^d$ we obtain measures λ_i on $\mathbb{Z}_p^d$, supported on $(1 + q\mathbb{Z}_p)^d$, such that, for the same m,

$$(b^{m-1} - 1)\zeta_{\mathfrak{a}_i}(m) = \mathcal{N}(\mathfrak{a}_i)^{-m} \int (x_1 \cdots x_d)^{-m}\, d\lambda_i\,. \tag{6.2}$$

For a fixed x in $1+q\mathbb{Z}_p$ the function $s \mapsto x^s$ is defined and analytic on $\mathbb{Z}_p$. Since, in general, $\mathcal{N}(\mathfrak{a}_i)$ will not be congruent to 1 modulo q, we do the following. For any integer l and any integral ideal $\mathfrak{a}$ of k that is coprime to p, the function $s \mapsto \omega_p(\mathfrak{a})^l \langle \mathcal{N}\mathfrak{a}\rangle^{-s}$ is an analytic $\mathbb{Z}_p$-valued function on $\mathbb{Z}_p$, whose value equals $\mathcal{N}(\mathfrak{a})^{-m}$ at any integer m with $m+l \equiv 0 \pmod{\varphi(q)}$. We define the p-adic partial ζ-function of the class of $\mathfrak{a}_i$ by mapping s in $\mathbb{Z}_p \setminus \{1\}$ to

$$\zeta_{p,\mathfrak{a}_i}(s) = \left(b^{s-1}-1\right)^{-1} \langle \mathcal{N}(\mathfrak{a}_i)\rangle^{-s} \int' (x_1 \cdots x_d)^{-s}\, d\lambda_i\,,$$

where $'$ indicates that we restricted the domain of integration to $(1+q\mathbb{Z}_p)^d$.

Let Ψ_i be the function defined by the above integral, that is, for s in $\mathbb{Z}_p$,

$$\Psi_i(s) = \int' (x_1 \cdots x_d)^{-s}\, d\lambda_i\,.$$

For a in $\mathbb{Z}_p$ with $|a-1|_p \leq q^{-1}$, and s in $\mathbb{Z}_p$, one can write $a^{-s} = \exp_p(-s\log_p(a))$, where $\exp_p$ and $\log_p$ are the p-adic exponential and logarithm functions respectively; and thus the function $s \mapsto a^{-s}$ can be expressed as a power series with coefficients in $\mathbb{Z}_p$, where the coefficient of s^m has absolute value at most $q^{-m}p^{m/(p-1)}$. Developing $(x_1 \cdots x_d)^{-s}$ in this way as a power series of s and using the fact that the measures λ_i have bounded norm by [18, Lemme 4.2 bis], we see that $\Psi_i(s)$ can also be expressed as a power series in $\mathbb{Q}_p[[s]]$ where the absolute value of the coefficient of s^m is at most $C_i q^{-m}p^{m/(p-1)}$ for some $C_i > 0$. In the same way, the function $s \mapsto \langle \mathcal{N}(\mathfrak{a}_i)\rangle^{-s}$ can be expressed as a power series in $\mathbb{Z}_p[[s]]$ whose coefficients satisfy the same bounds. Similarly, since the only solution for s of $b^{s-1}=1$ with $|s| < qp^{-1/(p-1)}$ is $s=1$, it follows from the p-adic Weierstrass preparation theorem that the function $s \mapsto b^{s-1}-1$ can be expressed as the product of a power of p, the polynomial $s-1$, and an invertible power series in $\mathbb{Z}_p[[s]]$ with the absolute value of the coefficient of s^m bounded by $q^{-m}p^{m/(p-1)}$. Therefore $\zeta_{p,\mathfrak{a}_i}(s)$ can be expressed as the quotient of a power series in $\mathbb{Q}_p[[s]]$ with the absolute value of the coefficient of s^m bounded by $C_i' q^{-m}p^{m/(p-1)}$ with $C_i' > 0$, and $s-1$.

If we define p-adic functions of s,

$$L_p^{(l)}(s,\chi,k) = \sum_{i=1}^{h} \omega_p(\mathfrak{a}_i)^l \chi(\mathfrak{a}_i)\zeta_{p,\mathfrak{a}_i}(s) \qquad (l \text{ modulo } \phi(q))\,, \tag{6.3}$$

then it follows from (6.1), (6.2), and the equality $\zeta_{p,\mathfrak{a}_i}(m) = \omega_p(\mathfrak{a}_i)^m \zeta_{\mathfrak{a}_i}(m)$ for any integer $m \leq 0$, that

$$L_p^{(l)}(m,\chi,k) = \mathrm{Eul}_p^*(m,\chi,k)L^*(m,\chi,k) \tag{6.4}$$

for all non-positive m such that $l+m \equiv 0$ modulo $\varphi(q)$. In particular, the p-adic L-function of χ is given by $L_p(s,\chi,k) = L_p^{(-1)}(s,\chi,k)$. Moreover, its residue at $s=1$ is zero if χ is non-trivial because the functions $\omega_p(\mathfrak{a}_i)^{-1}\zeta_{p,\mathfrak{a}_i}(s)$ all have the same residue there by [18, Corollaire on page 388].

The estimates on the coefficients of the power series expansions above imply the claims in Theorem 2.9(1). Through the p-adic Weierstrass preparation theorem they also imply that the function in (2.11) can be written as the quotient of a power series in $\mathbb{Q}_p(\chi)[[s]]$ that converges for s in $\mathbb{C}_p$ with $|s| < qp^{-1/(p-1)}$, and a polynomial in $\mathbb{Q}_p(\chi)[s]$.

We note in passing that if l is any integer then

$$L_p(s,\chi\omega_p^l,k)=\sum_{i=1}^{h}\omega_p(\mathfrak{a}_i)^{l-1}\chi(\mathfrak{a}_i)\zeta_{p,\mathfrak{a}_i}(s)=L_p^{(l-1)}(s,\chi,k)\,,$$

so that we obtain from (6.4) that

$$L_p(m,\chi\omega_p^l,k)=\mathrm{Eul}_p^*(m,\chi,k)L^*(m,\chi,k)$$

for all $m\le 0$ with $l+m\equiv 1$ modulo $\varphi(q)$, as stated in Remark 2.13.

In order to compute $L(s,\chi,k)$ for a fixed s in $\mathbb{Z}_p$, it suffices to compute $\zeta_{p,\mathfrak{a}_i}(s)$, for which one mainly has to compute the $\Psi_i(s)$. The measures λ_i can be represented as power series in d variables,

$$F_i(w_1,\dots,w_d)=\sum_{n_1,\dots,n_d\ge 0}a_{i,n_1,\dots,n_d}w_1^{n_1}\cdots w_d^{n_d}\,,$$

with (bounded) coefficients

$$a_{i,n_1,\dots,n_d}=\int\binom{x_1}{n_1}\cdots\binom{x_d}{n_d}d\lambda_i$$

that can be computed using Shintani's method [49], which is practical for calculations if the base field k is $\mathbb{Q}$ or real quadratic. On the other hand, the function h on $\mathbb{Z}_p$ defined by

$$x\mapsto\begin{cases}x^{-s} & \text{if } x \text{ is in } 1+q\mathbb{Z}_p,\\ 0 & \text{otherwise,}\end{cases}$$

is continuous and thus admits a Mahler expansion $\sum_{n\ge 0}c_{s,n}\binom{x}{n}$, where the coefficients satisfy $c_{s,n}\to 0$ when $n\to\infty$ and can be easily computed recursively. Then we have

$$\Psi_i(s)=\int h(x_1)\cdots h(x_d)\,d\lambda_i=\sum_{n_1,\dots,n_d\ge 0}a_{i,n_1,\dots,n_d}c_{s,n_1}\cdots c_{s,n_d}\,.$$

Thus, by computing enough terms in the above sum, we can get a good approximation of the integral, and therefore, putting everything together, of the p-adic L-function.

In order to apply this we make (2.10) explicit in the cases we are interested in, writing L-functions of representations for the L-functions of the corresponding characters. In particular, we write the classical L-functions of the irreducible 2-dimensional representations of S_3 and D_8 in Examples 5.3 and 5.5, and the irreducible 2-dimensional representations W_a with $a\ne 1$ of $S_3\times\mathbb{Z}/3\mathbb{Z}$ in Example 5.7, as the L-functions of Abelian characters over quadratic fields. For later use we also note how to write some of them in terms of ζ-functions of certain fields.

We take the field of coefficients E to be $\mathbb{C}$ in the remainder of this section, and denote the trivial 1-dimensional representation of any group G by 1_G.

Example 6.5. Let Alt_{S_3} be the 1-dimensional representation of S_3 through the sign character, and V the irreducible 2-dimensional representation. Since

$$\mathrm{Ind}_{A_3}^{S_3}\circ\mathrm{Ind}_{\{e\}}^{A_3}(1_{\{e\}})\cong\mathrm{Ind}_{\{e\}}^{S_3}(1_{\{e\}})\cong 1_{S_3}\oplus\mathrm{Alt}_{S_3}\oplus V^2$$

and $\mathrm{Ind}_{A_3}^{S_3}(1_{A_3})\cong 1_{S_3}\oplus\mathrm{Alt}_{S_3}$ it follows that $\mathrm{Ind}_{A_3}^{S_3}(V')\cong V$ for either non-trivial 1-dimensional representation V' of A_3. Applying this to the situation of Example 5.3 we see that $L(s,V,\mathbb{Q})=L(s,V',q)$. Similarly, if H is any subgroup of S_3 of

order 2, then $\mathrm{Ind}_H^{S_3}(1_H) \cong 1_{S_3} \oplus V$. In the situation of Example 5.3 this means that $\zeta_c(s) = \zeta_{\mathbb{Q}}(s)L(s, V, \mathbb{Q})$.

Example 6.6. If D_8 is as in Example 5.5, $V_{a,b}$ for $a, b = \pm 1$ the 1-dimensional representation of D_8 where σ acts as multiplication by a and τ by b, and V the irreducible 2-dimensional representation, then for any subgroup H

$$\mathrm{Ind}_H^{D_8} \circ \mathrm{Ind}_{\{e\}}^H(1_{\{e\}}) \cong (\oplus_{a,b=\pm 1} V_{a,b}) \oplus V^2 .$$

If we take $H = \langle \tau, \sigma^2 \rangle$ then $\mathrm{Ind}_{\{e\}}^H(1_{\{e\}}) \cong \oplus_{a,b=\pm 1} W_{a,b}$ with $W_{a,b}$ the 1-dimensional representation of H where σ^2 acts as multiplication by a and τ by b. Because $\mathrm{Ind}_H^{D_8}(W_{1,b}) \cong \oplus_{a=\pm 1} V_{a,b}$ we have $\mathrm{Ind}_H^{D_8}(W_{-1,1}) \cong \mathrm{Ind}_H^{D_8}(W_{-1,-1}) \cong V$, so with notation as in Figure 5.1 we get $L(s, V, \mathbb{Q}) = L(s, W_{-1,1}, q_1) = L(s, W_{-1,-1}, q_1)$.

Similarly, $\mathrm{Ind}_{\langle \tau \rangle}^{D_8}(1_{\langle \tau \rangle}) \cong (\oplus_{a=\pm 1} V_{a,1}) \oplus V$ so that $\zeta_{k_1}(s) = \zeta_{q_1}(s)L(s, V, \mathbb{Q})$ as well.

Remark 6.7. If $H = \langle \sigma \rangle$ in Example 6.6 then $\mathrm{Ind}_{\{e\}}^H(1_{\{e\}}) \cong \oplus_{m=1}^4 U_{i^m}$ with U_a the 1-dimensional representation of $\langle \sigma \rangle$ where σ acts as multiplication by a. Now $\mathrm{Ind}_H^{D_8}(U_a) \cong V_{a,1} \oplus V_{a,-1}$ when $a = \pm 1$ so that $\mathrm{Ind}_H^{D_8}(U_a) \cong V$ when $a \neq \pm 1$, and in Example 5.5 we also have $L(s, V, \mathbb{Q}) = L(s, U_i, q_2) = L(s, U_{-i}, q_2)$.

Example 6.8. If $G = S_3 \times \mathbb{Z}/3\mathbb{Z}$ and $H = A_3 \times \mathbb{Z}/3\mathbb{Z}$ then

$$\mathrm{Ind}_H^G \circ \mathrm{Ind}_{\{(e,0)\}}^H(1_{\{(e,0)\}}) \cong W \oplus W_1^2 \oplus W_{\zeta_3}^2 \oplus W_{\zeta_3^2}^2$$

with W the direct sum of the 1-dimensional representations of G and the W_a as in Example 5.7. Write $\mathrm{Ind}_{\{(e,0)\}}^H(1_{\{(e,0)\}}) \cong \oplus_{a,b} V_{a,b}$ where, for a and b cubic roots of unity, $V_{a,b}$ is the 1-dimensional representation of H on which (σ^m, n) acts as multiplication by $a^m b^n$. Since $\mathrm{Ind}_H^G(\oplus_b V_{1,b}) \cong W$ we find by considering the action of $(e, 1)$ that $\mathrm{Ind}_H^G(V_{a,b}) \cong W_b$ when $a \neq 1$. In particular, in the notation of Example 5.7, we have $L(s, W_b, \mathbb{Q}) = L(s, V_{a,b}, q)$ since $q = k^H$.

It is easy to see that the coefficients of p^{-s} in the Euler factors of $L(s, W_{\zeta_3}, \mathbb{Q})$ and $L(s, W_{\zeta_3^2}, \mathbb{Q})$ are conjugate, but since not all of them can be real it is not possible to express those functions in terms of ζ-functions of subfields of k.

7. Examples

In this section we describe the evidence for Conjecture 3.18 in the case of primitive idempotents π in $E[\mathrm{Gal}(k/\mathbb{Q})]$ as in Examples 5.3, 5.5 and 5.7, with $E = \mathbb{Q}$ in the first two and $E = \mathbb{Q}(\zeta_3)$ in the last. In the first two cases part (1) of the conjecture is known by Proposition 5.9, but in the last case our evidence for this part is numerical; the same holds for parts (2) through (4) in all cases.

We would like to stress that the numerical verification of part (4) of the conjecture actually proves this part for all cases that we considered since we can check that a element of $E \otimes \mathbb{Q}_p$ is a unit of that ring by computing a close enough approximation of it.

For the numerical calculations of the p-adic regulator, $R_{n,p}(M_\pi^E)$, we assumed that the syntomic regulator map on the subgroup of $K_{2n-1}(k)_{\mathbb{Q}}$ described in Zagier's Conjecture 4.5 is given by the $\widetilde{P}_{n,p}^{\sigma}$ (as in Remark 4.15). Out of the many possible choices for $P_{n,p}(z)$ as described in (4.12) we used

$$P_{n,p}(z) = \mathrm{Li}_{n,p}(z) - \log_p^{n-1}(z)\mathrm{Li}_1(z)/n! = \mathrm{Li}_{n,p}(z) + \log_p^{n-1}(z)\log_p(1-z)/n!$$

since it is relatively simple from a computational point of view. The calculations of the $P_{n,p}(z)$ for the branch with $\log_p(p) = 0$ were done in versions 2.11-7, 2.12-19 and 2.12-21 of [52], using an implementation of the algorithm described in [6].

Note that the primitive idempotents in Examples 5.3 and 5.5 are unique up to conjugation in $\mathbb{Q}[G]$ since they all correspond to π_2. But in Example 5.7 replacing ζ_3 with ζ_3^2 changes a primitive idempotent corresponding to π_3 into one corresponding to π_4, and conversely. Because the same holds for the classical and p-adic L-functions (with values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ and $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ respectively), as well as the Euler factor for p, the conjectures for primitive idempotents corresponding to π_3 and π_4 are actually equivalent.

In Examples 7.1, 7.4 and 7.5 we have $E = \mathbb{Q}$ so that we identify $E \otimes_{\mathbb{Q}} \mathbb{C}$ with $\mathbb{C}$, etc., and use notation as in Remark 3.17.

Example 7.1. Let k be a totally real S_3-extension of $\mathbb{Q}$. Use notation as in Case 1 of Example 5.3, and let π be the primitive idempotent corresponding to π_2 of that example. Since $E = \mathbb{Q}$ we shall write M_π instead of M_π^E in all notation referring to Conjecture 3.18.

Since $\pi K_{2n-1}(k)_{\mathbb{Q}} = \pi_2 K_{2n-1}(c)_{\mathbb{Q}}$ by Remark 3.22 we computed the K-theory of c using the (numerical) methods of Section 5 and tried to recognize the number

$$e(n, M_\pi) = L(n, \chi_\pi, \mathbb{Q}) \cdot D(M_\pi)^{1/2,\infty} / R_{n,\infty}(M_\pi) \tag{7.2}$$

for $n = 3$ and 5 as an element of $\mathbb{Q}^*$ by employing the same methods as described in Section 5 for number fields, where $L(n, \chi_\pi, \mathbb{Q})$ was computed as $\zeta_c(n)/\zeta_{\mathbb{Q}}(n)$ (see Example 6.5) using Pari-GP [53]. We succeeded in all cases, and then verified numerically for $p = 2$, 3, 5, 7 and 11 if $L_p(n, \chi_\pi \omega_p^{1-n}, \mathbb{Q}) \neq 0$ and

$$e(n, M_\pi) \cdot \mathrm{Eul}_p(n, \chi_\pi, \mathbb{Q}) \cdot \frac{R_{n,p}(M_\pi)}{D(M_\pi)^{1/2,p}} \cdot L_p(n, \chi_\pi \omega_p^{1-n}, \mathbb{Q})^{-1} \tag{7.3}$$

in $\mathbb{Q}_p^*$ was equal to 1.

We put our results for the four totally real S_3-extensions k of $\mathbb{Q}$ with smallest discriminants in Tables 1 through 4. Note that by Lemma 3.7 $R_{n,p}(M_\pi)/D(M_\pi)^{1/2,p}$ is in $\mathbb{Q}_p$ and the same holds for $L_p(n, \chi_\pi \omega_p^{1-n}, \mathbb{Q})$ by Lemma 3.15, so that (approximations of) those numbers can be easily represented in our tables. We have denoted an element $(a_0 + a_1 p + a_2 p^2 + \cdots) \times p^s$ in $\mathbb{Q}_p^*$ with all a_j in $\{0, \ldots, p-1\}$ and $a_0 \neq 0$ by $(a_0.a_1 a_2 \cdots)_p \times p^s$, writing A to represent 10 when $p = 11$.

For the p-adic regulators we computed each $P_{n,p}(z)$ up to $O(p^{L(p)})$ with $p^{L(p)}$ approximately equal to 10^{30}. The relevant part of $R_{n,p}(M_\pi)/D(M_\pi)^{1/2,p}$ has been given in the tables whenever it fitted. For the values of the p-adic L-function we can only prove that the relative error is at most $1 + O(p^{M(p)})$, where $M(2) = 72$, $M(3) = 47$, $M(5) = 32$, $M(7) = 26$ and $M(11) = 22$. This proves that $L_p(n, \chi_\pi \omega_p^{1-n}, \mathbb{Q})$ does not vanish; together with the verification that (7.2) equals 1 numerically this proves part (4) of Conjecture 3.18. The value that we found for (7.3) was $1 + O(p^{N(p)})$ with $N(2) \geq 82$, $N(3) \geq 52$, $N(5) \geq 36$, $N(7) \geq 27$ and $N(11) \geq 24$, giving numerical evidence for parts (2) and (3) of the conjecture, but also suggesting that the relative precision of the value of the p-adic L-function is slightly higher than we can prove, justifying the higher precision given in the tables.

Similar calculations were carried out for the next four such S_3 extensions and the same primes, with very similar results.

The other examples proceed mostly along the same lines as Example 7.1. The values of $L(n, \chi_\pi, \mathbb{Q})$ in Examples 7.4 and 7.5 below were computed using the formula at the end of Example 6.6, but in Example 7.6 the values of $L(n, \chi_\pi^E, \mathbb{Q})$, with $E = \mathbb{Q}(\zeta_3)$ for ζ_3 a primitive cube root of unity, were computed using the algorithms described in [24] and its associated program. In Examples 7.4 and 7.5 we determined if $e(n, \chi_\pi)$ (as in (7.2)) was in $\mathbb{Q}^*$ using the methods of Section 5, and for Example 7.6 we wrote $e(n, M_\pi^E)^{-1} = 1 \otimes a + \zeta_3 \otimes b$ in $E \otimes_{\mathbb{Q}} \mathbb{R}$ and then recognized a and b as integers divided by products of some relatively small primes. In each case we then verified for $p = 2, \ldots, 11$ if (7.3) (or its equivalent) was equal to 1 in $\mathbb{Q}_p$ or, for Example 7.6, in $\mathbb{Q}_p(\zeta_3) \otimes_{\mathbb{Q}} \mathbb{Q} = \mathbb{Q}_p(\zeta_3)$. The precision for $P_{n,p}(z)$ and the p-adic L-functions in Examples 7.4 and 7.5 was as in Example 7.1. In Example 7.6 all $P_{n,p}(z)$ were computed up to $O(p^{L(p)})$ with $p^{L(p)}$ approximately 10^{16} since in this case the check if the equivalent of (7.3) is 1 is done in $\mathbb{Q}(\zeta_3) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ so two coefficients are checked; the values of the p-adic L-function in this case, also in $\mathbb{Q}(\zeta_3) \otimes_{\mathbb{Q}} \mathbb{Q}_p$, were determined up to multiplication by $1 \otimes (1 + O(p^{M(p)}))$ with $M(p)$ as in Example 7.1.

Example 7.4. Let k be a totally real D_8-extension of $\mathbb{Q}$ as in Case 1 of Example 5.5, and use notation as in that example. We again write M_π instead of M_π^E since $E = \mathbb{Q}$ and consider odd $n \geq 2$. Because $\pi K_{2n-1}(k)_{\mathbb{Q}} = \pi_2 K_{2n-1}(k_1)_{\mathbb{Q}}$ we computed $K_{2n-1}(k_1)_{\mathbb{Q}}$ numerically using the methods of Section 4. Our results with $n = 3$ and 5 for the first four such extensions when ordered according to the discriminant are in Tables 5 through 8. The value that we found for (7.3) in this case was $1 + O(p^{N(p)})$ where $N(2) \geq 76$, $N(3) \geq 52$, $N(5) \geq 34$, $N(7) \geq 29$ and $N(11) \geq 22$. Such calculations were also carried out for the next four such extensions, with similar results.

Example 7.5. Let k be a CM Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(k/\mathbb{Q}) \cong D_8$ as in Case 2 of Example 5.5, and use notation as in that example. We again write M_π instead of M_π^E since $E = \mathbb{Q}$, and consider even $n \geq 2$. Again $\pi K_{2n-1}(k)_{\mathbb{Q}} = \pi_2 K_{2n-1}(k_1)_{\mathbb{Q}}$, so we computed $K_{2n-1}(k_1)_{\mathbb{Q}}$ numerically using the methods of Section 4. We put our results at $n = 2$ and 4 for the first four such extensions when ordered by the value of the (positive) discriminant in Tables 9 through 12. The value that we found for (7.3) in this case was $1 + O(p^{N(p)})$ where $N(2) \geq 79$, $N(3) \geq 53$, $N(5) \geq 35$, $N(7) \geq 27$ and $N(11) \geq 25$. Such calculations were also carried out for the next four such extensions, with similar results.

Example 7.6. Let k be a totally real $S_3 \times \mathbb{Z}/3\mathbb{Z}$ extension of $\mathbb{Q}$, as in Case 1 of Example 5.7. We use notation as in that example, taking $E = \mathbb{Q}(\zeta_3)$ for a primitive cubic root of unity ζ_3. A different identification of $\mathrm{Gal}(k/\mathbb{Q})$ with $S_3 \times \mathbb{Z}/3\mathbb{Z}$ might interchange π_3 and π_4 as well as corresponding primitive idempotents, but, as we noted before, the validity of the conjecture for either is equivalent.

Since $\pi K_{2n-1}(k)_E = \pi_3 K_{2n-1}(s_1)_E$ we computed $K_{2n-1}(s_1)_{\mathbb{Q}}$ as before. Also, in order to speed up the calculations of the regulator we note that the action of $(e, 1)$ and $(\sigma^2, 0)$ on $s_1 = k^{\langle(\sigma,1)\rangle}$ is the same, so that $\pi = \pi_3 \pi_{H_1}$ and

$$(1 - \pi_1)((e, 0) + \zeta_3(\sigma, 0) + \zeta_3^2(\sigma^2, 0))/3 = ((e, 0) + \zeta_3(\sigma, 0) + \zeta_3^2(\sigma^2, 0))/3$$

give the same result when applied to $K_{2n-1}(s_1)_E$. We put our results for the first four s_1, when ordered according to their discriminant, for $n = 3$ and 5 in Tables 13

through 16. In Table 16 the number $e(5, M_\pi^E)^{-1}$ is given by

$$-2^5 \cdot 3^{10} \cdot 5^{-2} \cdot 13^4 \cdot (7 + 3\zeta_3) \cdot (68 - 43\zeta_3) \cdot (1547202 - 1603487\zeta_3) \cdot (a + b\zeta_3)$$

with

$$a = 848225589231113909118621768674123 3$$

and

$$b = 352744050005881742101809475794761 7\,,$$

presumably because we experimentally found a subgroup of $K_{2n-1}(s_1)_{\mathbb{Q}}$ that is not close to actually being $K_{2n-1}(s_1)$. In all cases $L_p(n, \chi_\pi \otimes \omega_p^{1-n}, \mathbb{Q})$ is a unit in $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and when we write the equivalent of (7.3) in the form $1 \otimes \alpha + \zeta_3 \otimes \beta$ then we find that $\alpha - 1$ and β are $O(p^{N(p)})$ with $N(2) \geq 50$, $N(3) \geq 24$, $N(5) \geq 20$, $N(7) \geq 14$ and $N(11) \geq 12$, with in this case the precision bounded by that of the $R_{n,p}(M_\pi^E)$. Since $e(n, M_\pi^E)$, $D(M_\pi^E)^{1/2,p}$ and $\mathrm{Eul}_p(n, \chi_\pi \otimes \mathrm{id}, \mathbb{Q})$ are units in $E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$ this also shows that $R_{n,p}(M_\pi^E)$ is a unit in $E \otimes_{\mathbb{Q}} F$, as conjectured in Conjecture 3.18(4).

Such calculations were also carried out for the next four such s_1 with similar results.

Remark 7.7. By Proposition 4.17 and Remark 3.21(1), the conjecture for π in Example 7.1 is equivalent to Conjecture 2.17 for the field k_1 in the example. The same statement holds for π and k_1 in Example 7.4 below.

Remark 7.8. In the tables we have also included the value of the constant $C(\chi_\pi \otimes \mathrm{id}) = C(\chi_\pi^\vee \otimes \mathrm{id})$ in the functional equation (3.25) of $L(s, \chi_\pi \otimes \mathrm{id}, \mathbb{Q})$ to make possible a comparison between its prime factors and those of $e(n, M_\pi)$. Namely, for the $D(M_\pi^E)^{1/2,\infty}$ used in the calculations, the e_n in (3.26) (which in our case are independent of n) are as follows.

Table	e_n	Table	e_n	Table	e_n	Table	e_n
1	3	5	4	9	4	13	$1 \otimes 9 + \zeta_3 \otimes 9$
2	3	6	4	10	4	14	$1 \otimes 9 + \zeta_3 \otimes 9$
3	3	7	4	11	-4	15	$1 \otimes 0 + \zeta_3 \otimes 9$
4	3	8	4	12	4	16	$1 \otimes 0 + \zeta_3 \otimes (-9)$

As can be seen from these values and Tables 1-16, in the number

$$\delta_n e_n((n-1)!/2)^m C(\chi_\pi^\vee \otimes \mathrm{id})^{n-1} e(n, M_\pi^E)$$

of (3.27) the larger prime factors occurring in $e(n, M_\pi^E)$ are often cancelled by those of $C(\chi_\pi^\vee \otimes \mathrm{id})^{n-1}$. This suggests that the formulation of Beilinson's conjecture at $s = 1 - n$ normally involves simpler prime factors in E^* than the formulation at $s = n$ in Conjecture 3.18(1), with the remaining complicated factors in our examples quite possibly due to an awkward choice of basis for $K_{2n-1}(M_\pi^E)$.

References

[1] D. Barsky. Fonctions zêta p-adiques d'une classe de rayon des corps de nombres totalement réels. In *Groupe d'Etude d'Analyse Ultramétrique (5e année: 1977/78)*, pages Exp. No. 16, 23. Secrétariat Math., Paris, 1978.

[2] A. A. Beilinson. Higher regulators and values of L-functions. *J. Sov. Math.*, 30:2036–2070, 1985.

[3] A. Besser. Syntomic regulators and p-adic integration I: rigid syntomic regulators. *Israel Journal of Math.*, 120:291–334, 2000.

[4] A. Besser. Syntomic regulators and p-adic integration II: K_2 of curves. *Israel Journal of Math.*, 120:335–360, 2000.

[5] A. Besser and R. de Jeu. The syntomic regulator for the K-theory of fields. *Annales Scientifiques de l'École Normale Supérieure*, 36(6):867–924, 2003.

[6] A. Besser and R. de Jeu. $Li^{(p)}$-service? An algorithm for computing p-adic polylogarithms. To appear in *Mathematics of Computation*.

[7] A. Besser and C. Deninger. p-adic Mahler measures. *J. Reine Angew. Math.*, 517:19–50, 1999.

[8] F. Beukers. Irrationality of some p-adic L-values, 2006. Preprint available from http://arxiv.org/abs/math.NT/0603277.

[9] S. Bloch. Higher regulators, algebraic K-theory, and zeta functions of elliptic curves. Manuscript from 1978 ("Irvine notes"). Published as volume 11 of CRM Monographs Series by American Mathematical Society.

[10] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. ENS*, 4:235–272, 1974.

[11] A. Borel. Cohomologie de SL_n et valeurs de fonctions zêta aux points entiers. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(4):613–636, 1977. Errata at vol. 7, p. 373 (1980).

[12] J. I. Burgos Gil. *The regulators of Beilinson and Borel*, volume 15 of *CRM Monograph Series*. Amer. Math. Soc., Providence, RI, 2002.

[13] F. Calegari. Irrationality of certain p-adic periods for small p. *Int. Math. Res. Not.*, (20):1235–1249, 2005.

[14] P. Cassou-Noguès. p-adic L-functions for totally real number field. In *Proceedings of the Conference on p-adic Analysis (Nijmegen, 1978)*, volume 7806 of *Report*, pages 24–37. Katholieke Univ. Nijmegen, 1978.

[15] P. Cassou-Noguès. Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques. *Invent. Math.*, 51(1):29–59, 1979.

[16] R. Coleman. Dilogarithms, regulators, and p-adic L-functions. *Invent. Math.*, 69:171–208, 1982.

[17] R. Coleman and B. Gross. p-adic heights on curves. In *Algebraic number theory*, pages 73–81. Academic Press, Boston, MA, 1989.

[18] P. Colmez. Résidu en $s = 1$ des fonctions zêta p-adiques. *Invent. Math.*, 91(2):371–389, 1988.

[19] R. de Jeu. Zagier's conjecture and wedge complexes in algebraic K-theory. *Compositio Mathematica*, 96:197–247, 1995.

[20] P. Deligne. Valeurs de fonctions L et périodes d'intégrales. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 313–346. Amer. Math. Soc., Providence, R.I., 1979. With an appendix by N. Koblitz and A. Ogus.

[21] P. Deligne. Interprétation motivique de la conjecture de Zagier reliant polylogarithms et régulateurs. Preprint, 1991.

[22] P. Deligne and K. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[23] C. Deninger and A. J. Scholl. The Beĭlinson conjectures. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 173–209. Cambridge Univ. Press, Cambridge, 1991.

[24] T. Dokchitser. Computing special values of motivic L-functions. *Experiment. Math.*, 13(2):137–149, 2004.

[25] A. B. Goncharov. Polylogarithms and motivic Galois groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 43–96. Amer. Math. Soc., Providence, RI, 1994.

[26] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.

[27] A. B. Goncharov. Deninger's conjecture of L-functions of elliptic curves at $s = 3$. *J. Math. Sci.*, 81(3):2631–2656, 1996. Algebraic geometry, 4.
[28] R. Greenberg. On p-adic Artin L-functions. *Nagoya Math. J.*, 89:77–87, 1983.
[29] M. Gros. Régulateurs syntomiques et valeurs de fonctions L p-adiques. I. *Invent. Math.*, 99(2):293–320, 1990. With an appendix by Masato Kurihara.
[30] M. Gros. Régulateurs syntomiques et valeurs de fonctions L p-adiques. II. *Invent. Math.*, 115(1):61–79, 1994.
[31] U. Jannsen. Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives. In *Beĭlinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 305–372. Academic Press, Boston, MA, 1988.
[32] N. Katz. p-adic interpolation of real analytic Eisenstein series. *Ann. of Math. (2)*, 104(3):459–571, 1976.
[33] N. Katz. p-adic L-functions for CM fields. *Invent. Math.*, 49(3):199–297, 1978.
[34] N. Katz. Review of l-adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 21–30. Amer. Math. Soc., Providence, RI, 1994.
[35] T. Kubota and H.-W. Leopoldt. Eine p-adische Theorie der Zetawerte. I. Einführung der p-adischen Dirichletschen L-Funktionen. *J. Reine Angew. Math.*, 214/215:328–339, 1964.
[36] S. Lang. *Algebraic number theory*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont., 1970.
[37] B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986.
[38] B. Mazur and A. Wiles. Class fields of abelian extensions of $\mathbf{Q}$. *Invent. Math.*, 76(2):179–330, 1984.
[39] J. Neukirch. The Beĭlinson conjecture for algebraic number fields. In *Beĭlinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 193–247. Academic Press, Boston, MA, 1988.
[40] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.
[41] W. Nizioł. On the image of p-adic regulators. *Invent. Math.*, 127:375–400, 1997.
[42] B. Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198pp, 1995.
[43] D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory 1*, volume 341 of *Lecture Notes in Mathematics*, pages 179–198. Springer Verlag, Berlin, 1973.
[44] X.-F. Roblot. Computing values of p-adic L-functions of totally real number fields. In preparation.
[45] X.-F. Roblot and D. Solomon. Verifying a p-adic abelian Stark conjecture at $s = 1$. *J. Number Theory*, 107(1):168–206, 2004.
[46] P. Schneider. Introduction to the Beilinson Conjectures. In *Beilinson's Conjectures on Special Values of L-Functions*, pages 1–35. Academic Press, Boston, MA, 1988.
[47] A. J. Scholl. An introduction to Kato's Euler systems. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, pages 379–460. Cambridge Univ. Press, Cambridge, 1998.
[48] J.-P. Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972)*, pages 191–268. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.
[49] T. Shintani. On evaluation of zeta functions of totally real algebraic number fields at non-positive integers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 23(2):393–417, 1976.
[50] C. Soulé. On higher p-adic regulators. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 372–401. Springer Verlag, Berlin, 1981.
[51] A. A. Suslin. K_3 of a field, and the Bloch group. *Trudy Mat. Inst. Steklov.*, 183:180–199, 229, 1990. Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).
[52] The Magma group, Sydney. *Magma*. Available from `http://magma.maths.usyd.edu.au/`.

[53] The PARI Group, Bordeaux. *PARI/GP, version* `2.2.11`, 2005. Available from `http://pari.math.u-bordeaux.fr/`.
[54] D. Zagier. Polylogarithms, Dedekind Zeta Functions and the Algebraic K-theory of Fields. In *Arithmetic algebraic geometry (Texel, 1989)*, pages 391–430. Birkhäuser Boston, Boston, MA, 1991.

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er-Sheva 84105, Israel

Department of Pure Mathematics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, United Kingdom

Faculteit Exacte Wetenschappen, Afdeling Wiskunde, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

Institut Camille Jordan, Université de Lyon, Université Lyon 1, CNRS – UMR 5208, 43 blvd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France

TABLE 1. Splitting field of $x^3 - x^2 - 3x + 1$, where $C(\chi_\pi) = 2^2 \cdot 37$.

$n = 3$	$e(3, M_\pi)^{-1} = 2^{-2} \cdot 3^{-2} \cdot 37^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.000101010010000000100001011111001000101011101001100101110110011000000)_2 \times 2^0$
3	$(1.010012121202211021200001102121121100201201100121200211021212)_3 \times 3^4$
5	$(4.232014333021402310411334411010313100314 0)_5 \times 5^6$
7	$(6.354304301412412415450326016336635)_7 \times 7^6$
11	$(2.62161235A928A3423563A7888A)_{11} \times 11^6$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.000000101001110001000011001000111110101100001010011011100110001010 00)_2 \times 2^2$
3	$(1.120212200222100211011210001200200121111101101010220)_3 \times 3^0$
5	$(4.110400244402324422330241310400141 40)_5 \times 5^0$
7	$(5.235163632265012613625435331 10)_7 \times 7^0$
11	$(A.9542728A692401225487A278)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = 2^4 \cdot 3^2 \cdot 5^{-2} \cdot 37^4$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.000010101101100111010000111010010000011000001000100000110111100111 01)_2 \times 2^6$
3	$(1.200111022020102011202110020222210102001020101122110122022)_3 \times 3^{12}$
5	$(3.214210342400224303122204142313310423002)_5 \times 5^8$
7	$(1.210300440535545012312442660423 1)_7 \times 7^{10}$
11	$(1.2069AA607609242834655465)_{11} \times 11^{10}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.000001011110011100011110000101111101001010100010100010011001111000 11)_2 \times 2^2$
3	$(1.202212200211210200111021112002200221121202220100121 0)_3 \times 3^0$
5	$(3.344044221131211414204120401131303 42)_5 \times 5^0$
7	$(4.660015655136223166452626153 50)_7 \times 7^0$
11	$(1.1A26918431A09200099A81A7)_{11} \times 11^0$

TABLE 2. Splitting field of $x^3 - 6x - 2$, where $C(\chi_\pi) = 2^2 \cdot 3^3 \cdot 7$.

$n = 3$	$e(3, M_\pi)^{-1} = -2^{-2} \cdot 3^4 \cdot 7^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.100000011100001101010101011111010110010001011110101000010001011 11010)_2 \times 2^0$
3	$(2.112202001122000121011220210001122212200010212121102002 2012)_3 \times 3^4$
5	$(3.423230210022010143322244021343222420 2143)_5 \times 5^6$
7	$(4.254436360051001001352331 4315122)_7 \times 7^7$
11	$(5.481569541633A6875525A00911)_{11} \times 11^6$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.011111011100110011001101111001111000101000001001101111001011101 10010)_2 \times 2^2$
3	$(1.121100010121011012021221022022211101101102021011220)_3 \times 3^0$
5	$(2.130000201223020112214123221 40400441)_5 \times 5^0$
7	$(4.555101524162454051565455560 00)_7 \times 7^2$
11	$(1.14A21577033A19227344806A)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = -2^4 \cdot 3^{14} \cdot 5^{-2} \cdot 7^4$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.101011011000001011001011001001101011011111111000110011100110000011 111)_2 \times 2^6$
3	$(2.120122222011210012110212011212210211022212221020022201)_3 \times 3^{14}$
5	$(4.040210142044334334412003042243311012202)_5 \times 5^8$
7	$(3.240341533141502650211624311 0353)_7 \times 7^9$
11	$(7.160964096503643537518492)_{11} \times 11^{10}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.010101110000101110111111101010101111011001011010111110100101100010 111)_2 \times 2^2$
3	$(1.200200012110000121210100102002221102220112210011 1000)_3 \times 3^0$
5	$(4.443111023334421212124024103331443 43)_5 \times 5^0$
7	$(3.415260303634656620022624420 00)_7 \times 7^0$
11	$(2.687349321A27130485753530)_{11} \times 11^0$

TABLE 3. Splitting field of $x^3 - 4x - 1$, where $C(\chi_\pi) = 229$.

$n = 3$	$e(3, M_\pi)^{-1} = -2^{-9} \cdot 3^{-2} \cdot 229^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.011101001010100000011111001010010001101101010000010010111001111000 10)_2 \times 2^0$
3	$(2.012012211010121011021101220021012212100110002220200211021221)_3 \times 3^4$
5	$(4.343324143330113232200312324144102132 2342)_5 \times 5^6$
7	$(6.26240163456250535363306100451011 6)_7 \times 7^6$
11	$(3.6717A2040148108526534 2054)_{11} \times 11^7$
p	$L_p(3, \chi_\pi\omega_p^{-2}, \mathbb{Q})$
2	$(1.011000010110011011101000101110110010000001100000100001001101111111 01)_2 \times 2^3$
3	$(2.212021120211112210022001121111211100020201012002122 0)_3 \times 3^0$
5	$(3.034143322414334414042410124421044 12)_5 \times 5^0$
7	$(3.240330441325230344330545061 50)_7 \times 7^0$
11	$(9.21A61A480A10801760893710)_{11} \times 11^1$
$n = 5$	$e(5, M_\pi)^{-1} = -2^{-8} \cdot 3^6 \cdot 11 \cdot 229^4$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.111101101010100110110100010111001110011010110011011101110000000 10001)_2 \times 2^5$
3	$(2.222112021012010001201220011021022211112022112001020100)_3 \times 3^{16}$
5	$(4.341422314433322112400041022111342 0102)_5 \times 5^{10}$
7	$(2.561203660564626663526425105311 0)_7 \times 7^{10}$
11	$(8.5A9220278400758A0193AA6)_{11} \times 11^{11}$
p	$L_p(5, \chi_\pi\omega_p^{-4}, \mathbb{Q})$
2	$(1.010100001110101000100110100111001100100011010001011011001011001 00011)_2 \times 2^3$
3	$(2.112100022222000210222202101202022011021200021010220)_3 \times 3^0$
5	$(4.420111323421103323411303243444401 34)_5 \times 5^0$
7	$(1.250322451646564022365254355 00)_7 \times 7^0$
11	$(5.375705346AA0810959728302)_{11} \times 11^0$

TABLE 4. Splitting field of $x^3 - 6x^2 + 2$, where $C(\chi_\pi) = 2^2 \cdot 3^4 \cdot 5$.

$n = 3$	$e(3, M_\pi)^{-1} = 2^{-2} \cdot 3^6 \cdot 5$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.111100101100000010110111100101110100111001011110110110001111111 10001)_2 \times 2^0$
3	$(2.120020100000012102010221000201210120200202222221111002100)_3 \times 3^6$
5	$(3.343222244001442334444044112411123 1400414)_5 \times 5^4$
7	$(5.611651266622656555064166304513 241)_7 \times 7^6$
11	$(2.805A8344265760A69A2490828A)_{11} \times 11^6$
p	$L_p(3, \chi_\pi\omega_p^{-2}, \mathbb{Q})$
2	$(1.101000001001101111101111101011010010011101010111111001111011001 00001111)_2 \times 2^2$
3	$(1.112021112101201110220220201010021121011012222220 1210)_3 \times 3^0$
5	$(2.132423132330112011333331434120221 34)_5 \times 5^0$
7	$(3.001234421426135414126240635 10)_7 \times 7^0$
11	$(2.2725A17844929880A8412281)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = -2^5 \cdot 3^{18} \cdot 5^2 \cdot 324762301$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.000101000110110011100000111100100001000011011001111010001010010 01110)_2 \times 2^7$
3	$(1.020022001101111121111210121212021220120022112210200 1)_3 \times 3^{18}$
5	$(1.321100221113142104412032003401031 323110)_5 \times 5^7$
7	$(3.555306366530540565010504251354 6)_7 \times 7^{10}$
11	$(A.568607725325302A5663A746)_{11} \times 11^{10}$
p	$L_p(5, \chi_\pi\omega_p^{-4}, \mathbb{Q})$
2	$(1.101111010100110011110010000011111110001101101011000101000000000 01010)_2 \times 2^2$
3	$(1.000210111221212112100100220110012211121121011110211 0)_3 \times 3^0$
5	$(2.303341302131203224102401130234410 00)_5 \times 5^0$
7	$(4.620421011223326306561300624 20)_7 \times 7^0$
11	$(5.491153185997A48326862A46)_{11} \times 11^0$

TABLE 5. Splitting field of $x^4 - x^3 - 3x^2 + x + 1$, where $C(\chi_\pi) = 5 \cdot 29$.

$n = 3$	$e(3, M_\pi)^{-1} = 2^{-6} \cdot 3^{-1} \cdot 29^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1111101011110001110001100011000011100000111111110110100011100100 1001)_2 \times 2^2$
3	$(1.2012202111200101100102001202210121011011210112012122022102 12)_3 \times 3^5$
5	$(2.212214212241004322311344323242242111131 43)_5 \times 5^3$
7	$(2.260524146626624605001621634435032)_7 \times 7^6$
11	$(2.7469785544497786253 1A25026)_{11} \times 11^6$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.0101001010011100010100100011100110101110000000110011010111010001100)_2 \times 2^2$
3	$(1.1011210202202012010202221001211000021000202202200 21)_3 \times 3^0$
5	$(4.3021032442342201414242010334030233 0)_5 \times 5^0$
7	$(1.1635506314325135250654020103 4)_7 \times 7^0$
11	$(9.9A7720A981020A9A32071790)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = 2^{-4} \cdot 3^2 \cdot 5^2 \cdot 29^4$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.0110110100110110001100000000001110111011001000101000110011111110 1001)_2 \times 2^8$
3	$(1.0002011102111201211120012012011221112210212111112002202122)_3 \times 3^{12}$
5	$(1.0302040011301341231302414121023024103 21)_5 \times 5^7$
7	$(5.6213331545344431613600254300115)_7 \times 7^{10}$
11	$(9.98414459608789A37989AA86)_{11} \times 11^{10}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.0111110010110010111010100111001011001101101000101110111100000011 1001)_2 \times 2^2$
3	$(1.1111012010001010110000200000021011010010220212100 211)_3 \times 3^0$
5	$(4.0134142141114000202431203001011 0013)_5 \times 5^0$
7	$(4.1003213463255322224531352601 5)_7 \times 7^0$
11	$(8.6637671480756A9916284377)_{11} \times 11^0$

TABLE 6. Splitting field of $x^4 - 2x^3 - 3x^2 + 4x + 1$, where $C(\chi_\pi) = 2^2 \cdot 3^2 \cdot 11$.

$n = 3$	$e(3, M_\pi)^{-1} = 2^{-2} \cdot 3^2 \cdot 11^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1100111111111001100110110000110111101000000100011000100000111000001110)_2 \times 2^5$
3	$(1.1012221122001111200021222002100220020211212001011120222121 21)_3 \times 3^2$
5	$(1.3213042203212141302102134014142431100344)_5 \times 5^6$
7	$(1.4444342055331115320052535056156 56)_7 \times 7^6$
11	$(9.6161355514915399350567016 0)_{11} \times 11^5$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.1111010110100100110011100100110000101001011101110001101010100000 10001)_2 \times 2^4$
3	$(1.1221211120021010011021120022021022220202012220120020)_3 \times 3^0$
5	$(1.0312343242124131121002433324313 4200)_5 \times 5^0$
7	$(1.2051510450153516435550341 2253)_7 \times 7^0$
11	$(4.A19905A97667183A61256312)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = 2^7 \cdot 3^{10} \cdot 5^{-2} \cdot 11^7 \cdot 151 \cdot 1389251$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1010100010001100100111000000011001101000001101000110010010110010 0011)_2 \times 2^{16}$
3	$(2.0002001102001121012120002112100010010220202210201202112 0)_3 \times 3^{10}$
5	$(3.0431243401011420413024431314322224111 01)_5 \times 5^8$
7	$(3.1016042121151553442503432550426)_7 \times 7^{10}$
11	$(4.A244AA713986547114383)_{11} \times 11^{13}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.0101100111100010010010011011011100111001010110111010110000000 1001101)_2 \times 2^4$
3	$(1.2200111010220200001122120000000211211201112212022102)_3 \times 3^0$
5	$(4.3111122441240321001012341433341 4223)_5 \times 5^0$
7	$(2.6346365164142400632103101 4001)_7 \times 7^0$
11	$(2.054143779893294A023A2760)_{11} \times 11^1$

TABLE 7. Splitting field of $x^4 - 6x^2 + 6$, where $C(\chi_\pi) = 2^7 \cdot 3^2$.

$n = 3$	$e(3, M_\pi)^{-1} = -2^8 \cdot 3^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1111111011010111011000100100100010011101001101111100010101111010110)_2 \times 2^{10}$
3	$(2.022112012020121000012010222021210101120000211220112222110100)_3 \times 3^2$
5	$(2.041234240331110012313440303434111221114)_5 \times 5^7$
7	$(4.126126412601443066506634321456330)_7 \times 7^6$
11	$(1.508753A40028580A83265A6397)_{11} \times 11^6$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.001110011110101001100001100001111110000011011100010100001001101001 10)_2 \times 2^2$
3	$(1.110202012012211011012211100112220010211120222001022)_3 \times 3^0$
5	$(3.241024234210303223303123220410413 20)_5 \times 5^1$
7	$(3.534215531115524034525021353 51)_7 \times 7^0$
11	$(9.993168A73395501045A926A6)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = -2^{25} \cdot 3^{10} \cdot 5^{-1} \cdot 11 \cdot 37 \cdot 180097$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.010000111101010001111101100101011000001011001000011100110101001101 11)_2 \times 2^{27}$
3	$(1.201102100221212021202010210001202101100111112022222212 02)_3 \times 3^{10}$
5	$(4.133210102222133211431002443214124043 44)_5 \times 5^9$
7	$(1.654226004510431131501563554101 4)_7 \times 7^{10}$
11	$(4.35786968692892127976707)_{11} \times 11^{11}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.001001001011111110011010010111010111111110110100110100000111111010 001)_2 \times 2^2$
3	$(1.010122010121001211000012100222201201221212001021 22)_3 \times 3^0$
5	$(2.443342402131440333403303413310023 1)_5 \times 5^0$
7	$(2.436132313435556206640024133 00)_7 \times 7^0$
11	$(A.957125591551731675713855)_{11} \times 11^0$

TABLE 8. Splitting field of $x^4 - 6x^2 - 4x + 2$, where $C(\chi_\pi) = 2^5 \cdot 17$.

$n = 3$	$e(3, M_\pi)^{-1} = -2^2 \cdot 3^{-2} \cdot 17^2$
p	$R_{3,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.101110001101110111001100010111111100010111101000011000111000011 01001)_2 \times 2^7$
3	$(1.100112211001110222221002211101010200010100212200121012011 1212)_3 \times 3^4$
5	$(2.133013343140014311104134324304413234 1000)_5 \times 5^6$
7	$(1.251366021640656215554304534633666)_7 \times 7^6$
11	$(6.8781153571182A7A3A52578590)_{11} \times 11^6$
p	$L_p(3, \chi_\pi \omega_p^{-2}, \mathbb{Q})$
2	$(1.011001010000110111100001110111001001101101101110110001000001011 00110)_2 \times 2^5$
3	$(2.211111112101012212010222001012010220221010110202111 2)_3 \times 3^0$
5	$(2.030101312433213422442012402023132 43)_5 \times 5^0$
7	$(2.456535465253456230020126010 06)_7 \times 7^0$
11	$(1.5054539309942A8686113521)_{11} \times 11^0$
$n = 5$	$e(5, M_\pi)^{-1} = -2^{14} \cdot 3^2 \cdot 5^{-2} \cdot 17^4$
p	$R_{5,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.011010011001111000111000000111011000110111111111011011010000000 0100111)_2 \times 2^{19}$
3	$(1.002202120011120020011002210011010122010002120102002002 1001)_3 \times 3^{12}$
5	$(4.411222213224411421314440230014401411 02)_5 \times 5^9$
7	$(4.642565662100023263454226636 2653)_7 \times 7^{10}$
11	$(5.30669526A542413A15559106)_{11} \times 11^{10}$
p	$L_p(5, \chi_\pi \omega_p^{-4}, \mathbb{Q})$
2	$(1.100001110011001000010100000100010000111001111010110000000110111100 101)_2 \times 2^5$
3	$(2.110000112220022212212000000201120100012101012012 2102)_3 \times 3^0$
5	$(1.003134341314114344030301101032324 30)_5 \times 5^1$
7	$(4.514045415040523121524046132 55)_7 \times 7^0$
11	$(2.A43315061300727A39739622)_{11} \times 11^0$

TABLE 9. Splitting field of $x^4 - 2x^3 + 5x^2 - 4x + 2$, where $C(\chi_\pi) = 2^3 \cdot 17$.

$n = 2$	$e(2, M_\pi)^{-1} = -17$
p	$R_{2,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1111000001111100010001101010011101000100010111111100001011001000001)_2 \times 2^8$
3	$(1.200101010222011210111110122000000001202021102222201211201012 2)_3 \times 3^4$
5	$(2.012030204343424032021402204212031113234 21)_5 \times 5^4$
7	$(4.114030545466541102310610322536430 0)_7 \times 7^4$
11	$(9.074667649A5446686188082 8A41)_{11} \times 11^4$
p	$L_p(2, \chi_\pi \omega_p^{-1}, \mathbb{Q})$
2	$(1.100111000100010011100110100111011101111001011011000111010101 0011)_2 \times 2^6$
3	$(1.221102101120021000010101002122101120121020222012121 1)_3 \times 3^0$
5	$(4.330034031011124021030340431241230 20)_5 \times 5^0$
7	$(6.556552544003026022162661402 00)_7 \times 7^0$
11	$(4.84968AAA0466783629843316)_{11} \times 11^0$
$n = 4$	$e(4, M_\pi)^{-1} = 2^4 \cdot 3^2 \cdot 17^3$
p	$R_{4,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.011110101010111101001001100100110000011100011010111010000100110 10010)_2 \times 2^{15}$
3	$(2.020202022222222120111210002222000200022210022012202010022 11)_3 \times 3^{10}$
5	$(3.121141434441031100040041101320123332342)_5 \times 5^8$
7	$(4.301032463651051654014512630106 36)_7 \times 7^8$
11	$(9.8864321414815928A83132426)_{11} \times 11^8$
p	$L_p(4, \chi_\pi \omega_p^{-3}, \mathbb{Q})$
2	$(1.101010001110101011001100001100010010110010010010010100110010 10011101)_2 \times 2^7$
3	$(1.022011000010120100220122010111222011201221112221111 1)_3 \times 3^0$
5	$(4.241240033223300233403220232041124 10)_5 \times 5^0$
7	$(1.346311361241612062 56315631 3232)_7 \times 7^0$
11	$(6.484018160860423737043861)_{11} \times 11^0$

TABLE 10. Splitting field of $x^4 - x^3 + 3x^2 - 2x + 4$, where $C(\chi_\pi) = 5 \cdot 41$.

$n = 2$	$e(2, M_\pi)^{-1} = 2^{-3} \cdot 5 \cdot 41$
p	$R_{2,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.001001011010100011111101100010110101111100111001010110011101100 0)_2 \times 2^6$
3	$(2.002122011122021020212221110110211112101002220221001211111220 2)_3 \times 3^4$
5	$(4.241323021213432040340422344024214414 3010)_5 \times 5^4$
7	$(5.662266510026466025124145316522 0553)_7 \times 7^4$
11	$(2.0A380682536A50856633A5963AA)_{11} \times 11^4$
p	$L_p(2, \chi_\pi \omega_p^{-1}, \mathbb{Q})$
2	$(1.100011001000100101101101000110010001100011011001111110111111100000 00)_2 \times 2^5$
3	$(1.000011211000022110110010110022011212212110020011122 0)_3 \times 3^0$
5	$(3.313223242001103234442424241412040 00)_5 \times 5^1$
7	$(6.206422112314101056341031123 15)_7 \times 7^0$
11	$(4.8488929AA248277415458863)_{11} \times 11^0$
$n = 4$	$e(4, M_\pi)^{-1} = 2^{-5} \cdot 3^2 \cdot 5^3 \cdot 41^3$
p	$R_{4,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.011010110110110010011100010110110010011000010001000011110010111000 10)_2 \times 2^9$
3	$(2.122012210021111221111100021021221221000021112120002101022 20)_3 \times 3^{10}$
5	$(3.034332132412142321021313242120102244 00)_5 \times 5^8$
7	$(3.213414245316054651045104222140 51)_7 \times 7^8$
11	$(A.180567AA0545A59652909A250)_{11} \times 11^8$
p	$L_p(4, \chi_\pi \omega_p^{-3}, \mathbb{Q})$
2	$(1.111000011111001011100110001010000011000011001000010000111000001000 111)_2 \times 2^6$
3	$(1.211021010020211100000111212012102012020000020211110 2)_3 \times 3^0$
5	$(1.224142013121421301311123344301232 00)_5 \times 5^1$
7	$(6.662006052166436554346006064 23)_7 \times 7^0$
11	$(3.5059939560A2AA91106844A7)_{11} \times 11^0$

TABLE 11. Splitting field of $x^4 - 2x^3 + 6x^2 - 5x + 2$, where $C(\chi_\pi) = 13 \cdot 17$.

$n = 2$	$e(2, M_\pi)^{-1} = 2^{-3} \cdot 13 \cdot 17$
p	$R_{2,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.1110011101101111011000011011001101001000110001000000110110000000100011)_2 \times 2^7$
3	$(1.00111011010102222101020210100122002111011122111210020001 0011)_3 \times 3^5$
5	$(4.120023422244420112231001420210131140033 31)_5 \times 5^4$
7	$(6.006505153325032231255155634044 6112)_7 \times 7^4$
11	$(9.58602382412A681650341722097)_{11} \times 11^4$
p	$L_p(2, \chi_\pi \omega_p^{-1}, \mathbb{Q})$
2	$(1.0101000100000000000100000010101111111100110000100000110011111110 0100)_2 \times 2^6$
3	$(2.021212220000202120220021112210020000100101220120 2000)_3 \times 3^1$
5	$(2.123000040231104421411031120010442 00)_5 \times 5^0$
7	$(5.650356440554664462051066623 53)_7 \times 7^0$
11	$(6.38803A213738A731776470A4)_{11} \times 11^0$
$n = 4$	$e(4, M_\pi)^{-1} = -2^{-5} \cdot 3^2 \cdot 13^3 \cdot 17^3$
p	$R_{4,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.0101011001010010011010001111001101000101101000101000100111011110 1010)_2 \times 2^{10}$
3	$(1.1111010110110112222112002001211122102011200022210022220110)_3 \times 3^{11}$
5	$(2.3000410004401344130023314112311214303 12)_5 \times 5^8$
7	$(2.6324624136046311531523142422150 2)_7 \times 7^8$
11	$(A.49299953329A1125214A939A7)_{11} \times 11^8$
p	$L_p(4, \chi_\pi \omega_p^{-3}, \mathbb{Q})$
2	$(1.0010011101010000111001100001111111101101100010011001100101110000111 10)_2 \times 2^7$
3	$(1.2001110201210120022112022021221001220010010202222220)_3 \times 3^1$
5	$(4.3111103423122330322444030010210 1300)_5 \times 5^0$
7	$(3.2325064501066421123311415 6026)_7 \times 7^0$
11	$(6.722A829215274666128567 04)_{11} \times 11^0$

TABLE 12. Splitting field of $x^4 + 3x^2 - 6x + 6$, where $C(\chi_\pi) = 2^2 \cdot 3^2 \cdot 11$.

$n = 2$	$e(2, M_\pi)^{-1} = -2^{-1} \cdot 3^2 \cdot 11$
p	$R_{2,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.0001111111001101011110110011000110000001000100001001001100100011 0100)_2 \times 2^6$
3	$(2.0011210002222111210021022011011011002112020000011200012011 01)_3 \times 3^2$
5	$(4.3221104204411043111243423001113032204 0334)_5 \times 5^4$
7	$(6.4600034066060161654336311013633 00)_7 \times 7^5$
11	$(4.355421A545250233478 74A49229)_{11} \times 11^3$
p	$L_p(2, \chi_\pi \omega_p^{-1}, \mathbb{Q})$
2	$(1.1110100010010100001110011111111001110101100000110111101011110010 1100)_2 \times 2^5$
3	$(1.2022010222110011100120101011011011020010020011221012)_3 \times 3^0$
5	$(3.2201202020113301422324232302221 3400)_5 \times 5^0$
7	$(2.4331451155620656102245160 2200)_7 \times 7^1$
11	$(7.5148145379639767 28A46001)_{11} \times 11^0$
$n = 4$	$e(4, M_\pi)^{-1} = -2 \cdot 3^8 \cdot 11^3$
p	$R_{4,p}(M_\pi)/D(M_\pi)^{1/2,p}$
2	$(1.0001110100100000011111011100010100111111001101110100110100100111 0101)_2 \times 2^{11}$
3	$(2.0222010120001012000101112011222120000210000112021000101001)_3 \times 3^8$
5	$(4.2000241003224110043231114410411410 20122)_5 \times 5^8$
7	$(5.1253644025610330036433151202115 0)_7 \times 7^8$
11	$(6.6966231827671A91984A434A7)_{11} \times 11^7$
p	$L_p(4, \chi_\pi \omega_p^{-3}, \mathbb{Q})$
2	$(1.1011100010110101010011011010010010101100011111000000100101001100 01101)_2 \times 2^6$
3	$(1.0221110122220100011212020012222021100000010022110011)_3 \times 3^0$
5	$(3.4420010444313234102121403023111 2200)_5 \times 5^0$
7	$(4.4400354050022620036453402 1346)_7 \times 7^0$
11	$(5.766584290652809A28501199)_{11} \times 11^0$

TABLE 13. Splitting field of $x^6 - x^5 - 6x^4 + 7x^3 + 4x^2 - 5x + 1$, where $C(\chi_\pi \otimes \mathrm{id}) = 2^2 \cdot 5 \cdot 19$.

$n = 3$	$e(3, M_\pi^E)^{-1} = 3^{-2} \cdot 19^2$
p	$R_{3,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.010101011100101010)_2 \times 2^4) + \zeta_3 \otimes ((1.111100111000011101)_2 \times 2^2)$
3	$1 \otimes ((2.121122121210110121)_3 \times 3^4) + \zeta_3 \otimes ((1.100202121120121202)_3 \times 3^4)$
5	$1 \otimes ((4.402020001241122012)_5 \times 5^3) + \zeta_3 \otimes ((1.402102204113114001)_5 \times 5^4)$
7	$1 \otimes ((2.1404154326422162)_7 \times 7^6) + \zeta_3 \otimes ((5.1444362630564332)_7 \times 7^6)$
11	$1 \otimes ((6.9026829596822)_{11} \times 11^6) + \zeta_3 \otimes ((8.048680429402A)_{11} \times 11^6)$
p	$L_p(3, \chi_\pi \otimes \omega_p^{-2}, \mathbb{Q})$
2	$1 \otimes ((1.010111010101101000)_2 \times 2^4) + \zeta_3 \otimes ((1.111111110001001011)_2 \times 2^2)$
3	$1 \otimes ((1.120012101022112222)_3 \times 3^0) + \zeta_3 \otimes ((2.212202101110022110)_3 \times 3^0)$
5	$1 \otimes ((1.222321004420021231)_5 \times 5^0) + \zeta_3 \otimes ((1.123432442320044212)_5 \times 5^0)$
7	$1 \otimes ((6.215644553551135401)_7 \times 7^0) + \zeta_3 \otimes ((1.642236344041330514)_7 \times 7^0)$
11	$1 \otimes ((6.6A0036761338492361)_{11} \times 11^0) + \zeta_3 \otimes ((8.749011AA3694540145)_{11} \times 11^0)$
$n = 5$	$e(5, M_\pi^E)^{-1} = 2^7 \cdot 3^2 \cdot 5^2 \cdot 19^4 \cdot (37754 + 43673\zeta_3)$
p	$R_{5,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.000011100001001101)_2 \times 2^9) + \zeta_3 \otimes ((1.101010111001110110)_2 \times 2^9)$
3	$1 \otimes ((1.110110120010021222)_3 \times 3^{12}) + \zeta_3 \otimes ((2.020220021122220210)_3 \times 3^{12})$
5	$1 \otimes ((3.020013014242432200)_5 \times 5^7) + \zeta_3 \otimes ((1.220134043203242432)_5 \times 5^7)$
7	$1 \otimes ((4.151415510644 50)_7 \times 7^{10}) + \zeta_3 \otimes ((6.53120060244502)_7 \times 7^{10})$
11	$1 \otimes ((4.285500134)_{11} \times 11^{12}) + \zeta_3 \otimes ((2.A3934821906)_{11} \times 11^{10})$
p	$L_p(5, \chi_\pi \otimes \omega_p^{-4}, \mathbb{Q})$
2	$1 \otimes ((1.110010010001000101)_2 \times 2^4) + \zeta_3 \otimes ((1.111111100101010110)_2 \times 2^2)$
3	$1 \otimes ((1.211021010101222221)_3 \times 3^0) + \zeta_3 \otimes ((2.122212200211101011)_3 \times 3^0)$
5	$1 \otimes ((1.401222401123304240)_5 \times 5^0) + \zeta_3 \otimes ((1.432224122043203014)_5 \times 5^0)$
7	$1 \otimes ((2.441442356342000541)_7 \times 7^0) + \zeta_3 \otimes ((3.664531413660053644)_7 \times 7^0)$
11	$1 \otimes ((6.3A4A10A60052302518)_{11} \times 11^0) + \zeta_3 \otimes ((4.7AA8843572652457A4)_{11} \times 11^0)$

TABLE 14. Splitting field of $x^6 - x^5 - 8x^4 - x^3 + 12x^2 + 7x + 1$, where $C(\chi_\pi \otimes \mathrm{id}) = 2^2 \cdot 5 \cdot 31$.

$n = 3$	$e(3, M_\pi^E)^{-1} = 2^{-2} \cdot 3^{-2} \cdot 31^2 \cdot \zeta_3$
p	$R_{3,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.100100010111101000)_2 \times 2^2) + \zeta_3 \otimes ((1.111010100101111010)_2 \times 2^0)$
3	$1 \otimes ((1.221220200011201020)_3 \times 3^4) + \zeta_3 \otimes ((2.020111011200022220)_3 \times 3^4)$
5	$1 \otimes ((1.141130342002111122)_5 \times 5^3) + \zeta_3 \otimes ((4.342032011212400441)_5 \times 5^3)$
7	$1 \otimes ((4.4505556355566640)_7 \times 7^6) + \zeta_3 \otimes ((3.0610023664215014)_7 \times 7^6)$
11	$1 \otimes ((7.3700473650A19)_{11} \times 11^6) + \zeta_3 \otimes ((1.3659A52396880)_{11} \times 11^6)$
p	$L_p(3, \chi_\pi \otimes \omega_p^{-2}, \mathbb{Q})$
2	$1 \otimes ((1.101110001100101000)_2 \times 2^2) + \zeta_3 \otimes ((1.010100111010111101)_2 \times 2^4)$
3	$1 \otimes ((2.111112222201000112)_3 \times 3^0) + \zeta_3 \otimes ((1.120112022212020122)_3 \times 3^0)$
5	$1 \otimes ((4.321142113321234330)_5 \times 5^0) + \zeta_3 \otimes ((1.130134403242024422)_5 \times 5^0)$
7	$1 \otimes ((5.530006300632330553)_7 \times 7^0) + \zeta_3 \otimes ((2.200601212403230506)_7 \times 7^0)$
11	$1 \otimes ((2.00762927347972431A)_{11} \times 11^0) + \zeta_3 \otimes ((A.191A1780051997A650)_{11} \times 11^0)$
$n = 5$	$e(5, M_\pi^E)^{-1} = -2^6 \cdot 3^3 \cdot 5^2 \cdot 31^4 \cdot (2 + \zeta_3) \cdot (2 - \zeta_3) \cdot (7 - 15\zeta_3) \cdot (54 + 31\zeta_3)$
p	$R_{5,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.011100011000010010)_2 \times 2^8) + \zeta_3 \otimes ((1.011100000000101011)_2 \times 2^8)$
3	$1 \otimes ((2.202202022111202222)_3 \times 3^{14}) + \zeta_3 \otimes ((1.112002110200111001)_3 \times 3^{15})$
5	$1 \otimes ((3.141433024141141322)_5 \times 5^8) + \zeta_3 \otimes ((2.002014030014311244)_5 \times 5^7)$
7	$1 \otimes ((6.50135302661002)_7 \times 7^{10}) + \zeta_3 \otimes ((4.04356114536113)_7 \times 7^{10})$
11	$1 \otimes ((5.A3306A68921)_{11} \times 11^{10}) + \zeta_3 \otimes ((3.85A6650344A)_{11} \times 11^{10})$
p	$L_p(5, \chi_\pi \otimes \omega_p^{-4}, \mathbb{Q})$
2	$1 \otimes ((1.100110110011011000)_2 \times 2^2) + \zeta_3 \otimes ((1.110110100001011101)_2 \times 2^4)$
3	$1 \otimes ((2.201101211210020202)_3 \times 3^0) + \zeta_3 \otimes ((1.212021202011010201)_3 \times 3^0)$
5	$1 \otimes ((4.011013012410044000)_5 \times 5^0) + \zeta_3 \otimes ((1.431333301302323324)_5 \times 5^0)$
7	$1 \otimes ((2.130552365243450401)_7 \times 7^0) + \zeta_3 \otimes ((2.601221155363314062)_7 \times 7^0)$
11	$1 \otimes ((9.53274A487A14818064)_{11} \times 11^0) + \zeta_3 \otimes ((9.89A625650825167861)_{11} \times 11^0)$

TABLE 15. Splitting field of $x^6 - x^5 - 8x^4 + 5x^3 + 19x^2 - 4x - 11$, where $C(\chi_\pi \otimes \mathrm{id}) = 5 \cdot 139$.

$n = 3$	$e(3, M_\pi^E)^{-1} = 2^{-5} \cdot 3^{-2} \cdot 139^2$
p	$R_{3,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.000110100111100100)_2 \times 2^4) + \zeta_3 \otimes ((1.000011011101110010)_2 \times 2^5)$
3	$1 \otimes ((2.102212010212021001)_3 \times 3^4) + \zeta_3 \otimes ((2.122111112211000220)_3 \times 3^4)$
5	$1 \otimes ((1.400211310114320013)_5 \times 5^3) + \zeta_3 \otimes ((1.434230430334022423)_5 \times 5^3)$
7	$1 \otimes ((2.2523154003106061)_7 \times 7^6) + \zeta_3 \otimes ((2.3200021260361110)_7 \times 7^6)$
11	$1 \otimes ((9.4472801A61A44)_{11} \times 11^6) + \zeta_3 \otimes ((1.7232221727046)_{11} \times 11^6)$
p	$L_p(3, \chi_\pi \otimes \omega_p^{-2}, \mathbb{Q})$
2	$1 \otimes ((1.000100101010000101)_2 \times 2^4) + \zeta_3 \otimes ((1.000010100000000110)_2 \times 2^3)$
3	$1 \otimes ((2.111012101221111220)_3 \times 3^2) + \zeta_3 \otimes ((1.010011111201220212)_3 \times 3^0)$
5	$1 \otimes ((3.240243311024013404)_5 \times 5^0) + \zeta_3 \otimes ((4.142342004244233331)_5 \times 5^2)$
7	$1 \otimes ((6.543343430631523116)_7 \times 7^1) + \zeta_3 \otimes ((2.342260265363120032)_7 \times 7^0)$
11	$1 \otimes ((4.8521241A647A883282)_{11} \times 11^0) + \zeta_3 \otimes ((1.46220220356163464A)_{11} \times 11^0)$
$n = 5$	$e(5, M_\pi^E)^{-1} = 2^{-4} \cdot 3^2 \cdot 5^2 \cdot 139^4 \cdot (4 - 5\zeta_3)$
p	$R_{5,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.000100111100110101)_2 \times 2^{10}) + \zeta_3 \otimes ((1.011111001011110010)_2 \times 2^9)$
3	$1 \otimes ((1.202001002112012221)_3 \times 3^{13}) + \zeta_3 \otimes ((1.021221211222021020)_3 \times 3^{12})$
5	$1 \otimes ((3.404332041431011004)_5 \times 5^7) + \zeta_3 \otimes ((3.213210304101144004)_5 \times 5^7)$
7	$1 \otimes ((4.11031052300040)_7 \times 7^{10}) + \zeta_3 \otimes ((4.03411453000015)_7 \times 7^{10})$
11	$1 \otimes ((6.8523188534)_{11} \times 11^{11}) + \zeta_3 \otimes ((5.2A39A447198)_{11} \times 11^{10})$
p	$L_p(5, \chi_\pi \otimes \omega_p^{-4}, \mathbb{Q})$
2	$1 \otimes ((1.101001011001000000)_2 \times 2^4) + \zeta_3 \otimes ((1.000100011110101010)_2 \times 2^3)$
3	$1 \otimes ((1.200212010102010102)_3 \times 3^1) + \zeta_3 \otimes ((1.122011112111021002)_3 \times 3^0)$
5	$1 \otimes ((3.124113213204101323)_5 \times 5^0) + \zeta_3 \otimes ((2.114244212313224030)_5 \times 5^3)$
7	$1 \otimes ((6.430024313466045624)_7 \times 7^0) + \zeta_3 \otimes ((5.065261416212661354)_7 \times 7^0)$
11	$1 \otimes ((5.A88931A0106A0172A3)_{11} \times 11^0) + \zeta_3 \otimes ((3.A4151A485571789978)_{11} \times 11^0)$

TABLE 16. Splitting field of $x^6 - x^5 - 8x^4 + 11x^3 + 2x^2 - 5x + 1$, where $C(\chi_\pi \otimes \mathrm{id}) = 2^2 \cdot 3^2 \cdot 13$.

$n = 3$	$e(3, M_\pi^E)^{-1} = -2^{-2} \cdot 3^2 \cdot 13^2 \cdot \zeta_3$
p	$R_{3,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.101100110101111100)_2 \times 2^2) + \zeta_3 \otimes ((1.001000001010110111)_2 \times 2^0)$
3	$1 \otimes ((2.021221201100120120)_3 \times 3^5) + \zeta_3 \otimes ((2.020012210111121212)_3 \times 3^7)$
5	$1 \otimes ((3.000431031211204434)_5 \times 5^6) + \zeta_3 \otimes ((1.012314421212444244)_5 \times 5^6)$
7	$1 \otimes ((3.0426445363200511)_7 \times 7^6) + \zeta_3 \otimes ((1.1035504615611621)_7 \times 7^6)$
11	$1 \otimes ((8.9799A36158285)_{11} \times 11^6) + \zeta_3 \otimes ((9.229923AAA0734)_{11} \times 11^6)$
p	$L_p(3, \chi_\pi \otimes \omega_p^{-2}, \mathbb{Q})$
2	$1 \otimes ((1.100100111001100011)_2 \times 2^2) + \zeta_3 \otimes ((1.101001000011101110)_2 \times 2^4)$
3	$1 \otimes ((1.022110110201121122)_3 \times 3^2) + \zeta_3 \otimes ((2.102110200120211001)_3 \times 3^0)$
5	$1 \otimes ((2.424401443414003124)_5 \times 5^0) + \zeta_3 \otimes ((4.440233034214103134)_5 \times 5^0)$
7	$1 \otimes ((5.421026022055251515)_7 \times 7^0) + \zeta_3 \otimes ((4.510213122044500212)_7 \times 7^0)$
11	$1 \otimes ((A.18743500687825560 8)_{11} \times 11^0) + \zeta_3 \otimes ((6.47020389393AA07A54)_{11} \times 11^0)$
$n = 5$	$e(5, M_\pi^E)^{-1} =$ see Example 7.6
p	$R_{5,p}(M_\pi^E)/D(M_\pi^E)^{1/2,p}$
2	$1 \otimes ((1.001111100010111111)_2 \times 2^7) + \zeta_3 \otimes ((1.100110001101001011)_2 \times 2^8)$
3	$1 \otimes ((1.100222220001012001)_3 \times 3^{15}) + \zeta_3 \otimes ((1.201100102012122102)_3 \times 3^{15})$
5	$1 \otimes ((1.302021433120040233)_5 \times 5^8) + \zeta_3 \otimes ((1.414010322010320220)_5 \times 5^8)$
7	$1 \otimes ((6.54022035336014)_7 \times 7^{10}) + \zeta_3 \otimes ((1.31434160250212)_7 \times 7^{10})$
11	$1 \otimes ((9.5A062420171)_{11} \times 11^{10}) + \zeta_3 \otimes ((A.873665790A8)_{11} \times 11^{10})$
p	$L_p(5, \chi_\pi \otimes \omega_p^{-4}, \mathbb{Q})$
2	$1 \otimes ((1.101111101101011100)_2 \times 2^2) + \zeta_3 \otimes ((1.001110000011000011)_2 \times 2^4)$
3	$1 \otimes ((2.001012111110022002)_3 \times 3^1) + \zeta_3 \otimes ((2.011221210002201001)_3 \times 3^0)$
5	$1 \otimes ((3.241120123211444141)_5 \times 5^1) + \zeta_3 \otimes ((2.103430110032401010)_5 \times 5^0)$
7	$1 \otimes ((5.454116060262152231)_7 \times 7^0) + \zeta_3 \otimes ((5.160242506230053303)_7 \times 7^0)$
11	$1 \otimes ((9.9950088486A6479846)_{11} \times 11^0) + \zeta_3 \otimes ((A.178492061888054088)_{11} \times 11^0)$