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The Inventory inaccuracy issue under a multiplicative error structure

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The notations used in this paper are as follows:

- D : the random variable representing demand
- $f(F)$: the PDF (CDF respectively) characterizing D
- μ_D : the average of D
- σ_D : the standard deviation of D
- $\gamma_{IS}(\gamma_{PH})$: the random variable representing IS (PH) errors
- $\mu_{IS}(\mu_{PH})$: the average of γ_{IS} (γ_{PH})
- $\sigma_{IS}(\sigma_{PH})$: the standard deviation of γ_{IS} (γ_{PH})
- $p(P)$: the PDF (CDF respectively) characterizing γ_{IS}
- $h(H)$: the PDF (CDF respectively) characterizing γ_{PH}
- r : the unit selling price
- c : the unit purchase cost
- s : the unit salvage cost
- K : the unit cost paid for a non-satisfied commitment
- Q : the ordering quantity
- Q^* : the optimal value of Q
- Π : the expected profit function

We consider a single-product and single-period (newsvendor) problem with the following sequence of events:

- Let Q be the quantity that the inventory manager orders from his supply system.
- He will receive goods, update the information system by scanning products and store them in the warehouse.
- Because of errors, $Q_{PH} = \gamma_{PH}Q$, i.e. what is physically available in inventory, may not be equal to $Q_{IS} = \gamma_{IS}Q$, i.e. what the information system shows as being available.
- Just before the beginning of the selling period, the inventory manager will receive a cumulative online order from the final customers. He will compare the total quantity requested by the customers with the IS inventory record to accept or decline orders. If the cumulative order is less than Q_{IS} , the e-retailer will accept all the orders. If not, he will only accept orders summing up to the IS inventory.
- Later on, products will be shipped to the customers. All the orders that the e-retailer has committed himself to should in principle be satisfied. However, this may not always be the case due to inventory inaccuracies. Such a situation occurs for instance when Q_{IS} is larger than Q_{PH} , and the demand is larger than Q_{IS} .

The expected profit formulation

Based on the sequence of events previously described the commitment C and the period sales could be written as follows:

$$C = \min(D, Q_{IS})$$
$$\text{Sales} = \min(C, Q_{PH})$$

For a given vector (D, Q_{IS}, Q_{PH}) , the profit achieved by the inventory manager could be deduced as follows:

$$\text{Profit} = r \cdot \text{Sales} + s \cdot (Q_{PH} - \text{Sales})^+ - c \cdot Q_{PH} - K \cdot (C - Q_{PH})^+$$

The first and the second terms of the profit correspond to the margin achieved from sales and salvages respectively. The third term is associated with purchase costs and the last one corresponds to the penalty paid when a commitment is not satisfied, i.e., when $C > Q_{PH}$. The profit function could also be written as follows:

$$\text{Profit} = r\text{Min}[\text{Min}(Q_{IS}, D), Q_{PH}] + s[Q_{PH} - \text{Min}(Q_{IS}, D)]^+ - cQ_{PH} - P[\text{Min}(Q_{IS}, D) - Q_{PH}]^+$$

After some basic algebra, the profit could be rewritten as:

$$\text{Profit} = (r - c)D - (r - c)[D - Q_{IS}]^+ - (r - c + P)\{(Q_{IS} - Q_{PH}) - \text{Min}[(Q_{IS} - D)^+, (Q_{IS} - Q_{PH})]\} + (c - s)\text{Min}[(Q_{IS} - D)^+, (Q_{IS} - Q_{PH})] - (c - s)[Q_{IS} - D]^+$$

The previous expression of the profit could be interpreted as follows:

$(r - c)D$: represents the expected sales revenue

$-(c - s)[Q_{IS} - D]^+ - (r - c)[D - Q_{IS}]^+ + (c - s)\text{Min}[(Q_{IS} - D)^+, (Q_{IS} - Q_{PH})]$: represents the cost paid in an overstocking situation

$-(r - c)[D - Q_{IS}]^+$: represents the underage penalty incurred if a demand is not satisfied when answering customers' requests

$-(r - c + K)\{(Q_{IS} - Q_{PH}) - \text{Min}[(Q_{IS} - D)^+, (Q_{IS} - Q_{PH})]\}$: represents the underage penalty incurred when a commitment is made and then not respected

Based on the previous expression explanation, we could define the following unit costs:

$u_1 = r - c$, the unit type 1 shortage cost paid when the IS level is not enough to satisfy a demand

$u_2 = r - c + P$, the unit type 2 shortage cost paid when the PH level is not enough to satisfy a commitment

$hol = r - c$, the unit overage cost

In the classical inventory literature, only hol and u_1 are considered since the IS level is implicitly assumed to be equal to the PH inventory level.

Considering the multiplicative error structure ($Q_{IS} = \gamma_{IS}Q$ and $Q_{PH} = \gamma_{PH}Q$) and applying the expectation to profit function permits us to deduce $\pi(Q)$ as follows:

$$\begin{aligned} \pi(Q) = & (r-c)\mu_D - (c-s) \int_{\gamma_{IS}=0}^{+\infty} \int_{D=-\infty}^{Q\gamma_{IS}} (\gamma_{IS}Q - D)f(D)p(\gamma_{IS}) dD d\gamma_{IS} - (r-c) \int_{\gamma_{IS}=0}^{+\infty} \int_{D=Q\gamma_{IS}}^{+\infty} (D - \gamma_{IS}Q)f(D)p(\gamma_{IS}) dD d\gamma_{IS} \\ & - (r-c+K)Q \int_{\gamma_{PH}=0}^{+\infty} \int_{\gamma_{IS}=0}^{+\infty} (\gamma_{IS} - \gamma_{PH})p(\gamma_{IS})h(\gamma_{PH}) d\gamma_{IS} d\gamma_{PH} \\ & + (r-s+K) \left(\int_{\gamma_{PH}=0}^{+\infty} \int_{\gamma_{IS}=0}^{+\infty} (\gamma_{IS} - \gamma_{PH})Qp(\gamma_{IS})h(\gamma_{PH}) d\gamma_{IS} d\gamma_{PH} \right. \\ & \left. + \int_{\gamma_{PH}=0}^{+\infty} \int_{\gamma_{IS}=\gamma_{PH}}^{+\infty} \int_{D=Q\gamma_{PH}}^{Q\gamma_{IS}} \left(\gamma_{IS} - \frac{D}{Q} \right) Qf(D)p(\gamma_{IS}) h(\gamma_{PH}) dD d\gamma_{IS} d\gamma_{PH} \right. \\ & \left. + \int_{\gamma_{PH}=0}^{+\infty} \int_{\gamma_{IS}=\gamma_{PH}}^{+\infty} \int_{D=Q\gamma_{PH}}^{Q\gamma_{IS}} (\gamma_{IS} - \gamma_{PH})Qf(D)p(\gamma_{IS}) h(\gamma_{PH}) dD d\gamma_{IS} d\gamma_{PH} \right) \end{aligned}$$

After some simplification of profit equation we obtain the final expression

$$\begin{aligned}
\pi(Q) = & u_1 \left(\mu_D - \mu_D(1 - P(0)) \right) - Q \int_{Y_{IS}=0}^{+\infty} Y_{IS} ((u_1 + \text{hol})F(Y_{IS}Q) - u_1)p(Y_{IS}) dY_{IS} \\
& + \int_{Y_{IS}=0}^{+\infty} (\mu_D(\text{hol} + u_1)F(Y_{IS}Q) - \sigma_D^2(\text{hol} + u_1)f(Y_{IS}Q))p(Y_{IS}) dY_{IS} - u_2Q(\mu_{IS} - \mu_{PH}) \\
& + (u_2 + \text{hol})Q \left(\int_{Y_{PH}=0}^{+\infty} ((\mu_{IS} - Y_{PH})(P(Y_{PH}) - P(0)) - \sigma_{IS}^2(p(Y_{PH}) - p(0)))h(Y_{PH}) dY_{PH} \right. \\
& + \int_{Y_{IS}=0}^{+\infty} \left(\left(Y_{IS}F(QY_{IS}) - \frac{\mu_D}{Q}F(QY_{IS}) + \frac{\sigma_D^2}{Q}f(QY_{IS}) \right) (H(Y_{IS}) - H(0)) \right) * p(Y_{IS}) dY_{IS} \\
& \left. - \int_{Y_{PH}=0}^{+\infty} \int_{Y_{PH}=0}^{Y_{IS}} \left((Y_{PH} - \frac{\mu_D}{Q})F(Y_{PH}Q) + \frac{\sigma_D^2}{Q}f(QY_{PH}) \right) h(Y_{PH})p(Y_{IS}) * dY_{PH} dY_{IS} \right)
\end{aligned}$$

The optimal inventory policy

The concavity of the expected profit function is not always verified. Next we provide the conditions of the existence of an optimal ordering policy permitting to minimize the expected profit.

Let's first defined three constants $C1$, $C2$ and $C3$ that could be easily calculated if the costs as well as the errors' parameters (average and standard deviation) are used:

$$\begin{aligned}
C1 = & (u_2 + \text{hol}) \int_0^{+\infty} \{ (Y_{IS} - \mu_{PH})H(Y_{IS}) + \sigma_{PH}^2 h(Y_{IS}) \} p(Y_{IS}) dY_{IS} \\
& + (u_2 + \text{hol})(1 - P(0)) (\mu_{PH}H(0) - \sigma_{PH}^2 h(0)) \\
& - (\text{hol} + (u_2 + \text{hol})H(0)) (\mu_{IS}(1 - P(0)) + p(0)\sigma_{IS}^2) \\
& - u_1(\mu_{IS}(1 - P(0)) + p(0)\sigma_{IS}^2)
\end{aligned}$$

$$\begin{aligned}
C2 = & -u_1(\mu_{IS}(1 - P(0)) + p(0)\sigma_{IS}^2) + u_2(\mu_{IS} - \mu_{PH}) \\
& - (u_2 + \text{hol}) \int_0^{+\infty} \left[((\mu_{IS} - Y_{PH})P(Y_{PH}) - \sigma_{IS}^2 p(Y_{PH})) h(Y_{PH}) dY_{PH} \right] \\
& - (u_2 + \text{hol})(1 - H(0)) (-\mu_{IS}P(0) + \sigma_{IS}^2 p(0)) \\
& - (u_2 + \text{hol})P(0) (\sigma_{PH}^2 h(0) + \mu_{PH}(1 - H(0)))
\end{aligned}$$

$C3$ solves the following equation

$$\begin{aligned}
\frac{\partial^2 \pi}{\partial Q^2} = & - \int_0^{+\infty} \left[(Y_{IS}^2 (u_1 + \text{hol})f(Y_{IS}C3)) p(Y_{IS}) dY_{IS} \right] \\
& + (u_2 + \text{hol}) \int_0^{+\infty} \left[\left[\int_0^{Y_{IS}} (-Y_{PH}^2 f(Y_{PH}C3)) h(Y_{PH}) dY_{PH} \right] p(Y_{IS}) dY_{IS} \right] \\
& + (u_2 + \text{hol}) \int_0^{+\infty} \left[(Y_{IS}^2 f(Y_{IS}C3)(H(Y_{IS}) - H(0))) p(Y_{IS}) dY_{IS} \right] = 0
\end{aligned}$$

Calculating the first derivative of the expected profit function and setting it equal to zero permits us to deduce that the optimal ordering quantity Q^* , as well as the conditions of its existence, is as it is stated in the following theorem:

Theorem:

The optimal ordering quantity maximizing the expected profit function when errors are taken into account should satisfy (1):

$$\int_0^{+\infty} \int_0^{Y_{IS}} \left[\begin{pmatrix} (u_2 + \text{hol})Y_{IS}F(Y_{IS}Q^*) \\ -(u_2 + \text{hol})Y_{PH}F(Y_{PH}Q^*) \\ -(u_1 + \text{hol})\frac{Y_{IS}F(Y_{IS}Q^*)}{H(Y_{IS}) - H(0)} \end{pmatrix} h(Y_{PH}) \right] p(Y_{IS}) dY_{PH} dY_{IS} = C2 \quad (1)$$

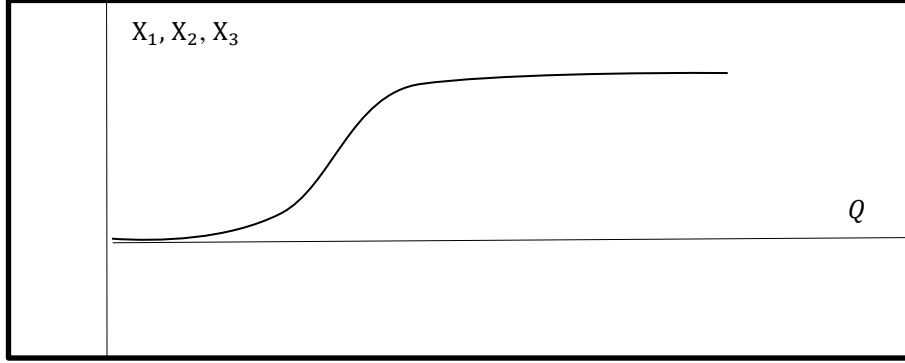
We decomposed this equation on three parts (X_1 , X_2 , X_3):

$$X_1(Q) = \int_0^{+\infty} \int_0^{\gamma_{IS}} [((u_2 + \text{hol})\gamma_{IS}F(\gamma_{IS}Q))h(\gamma_{PH})] p(\gamma_{IS})d\gamma_{PH}d\gamma_{IS}$$

$$X_2(Q) = \int_0^{+\infty} \int_0^{\gamma_{IS}} \left[\left((u_1 + \text{hol}) \frac{\gamma_{IS}F(\gamma_{IS}Q)}{H(\gamma_{IS}) - H(0)} \right) h(\gamma_{PH}) \right] p(\gamma_{IS})d\gamma_{PH}d\gamma_{IS}$$

$$X_3(Q) = \int_0^{+\infty} \int_0^{\gamma_{IS}} [((u_2 + \text{hol})\gamma_{PH}F(\gamma_{PH}Q))h(\gamma_{PH})] p(\gamma_{IS})d\gamma_{PH}d\gamma_{IS}$$

The behaviors of X_1 , X_2 and X_3 are similar to the cumulative function F (fig.1)



Fig;1 The behaviors of X_1 , X_2 and X_3

So the equation (1) will have this form: $X_1(Q) - X_2(Q) - X_3(Q) = C2$ with

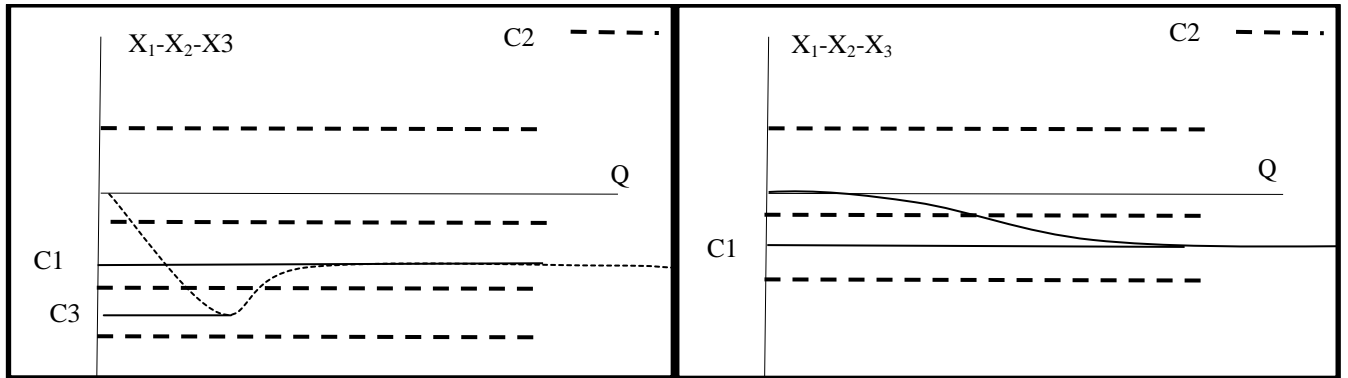
$$\lim_{Q \rightarrow 0} : X_1(Q) - X_2(Q) - X_3(Q) = 0$$

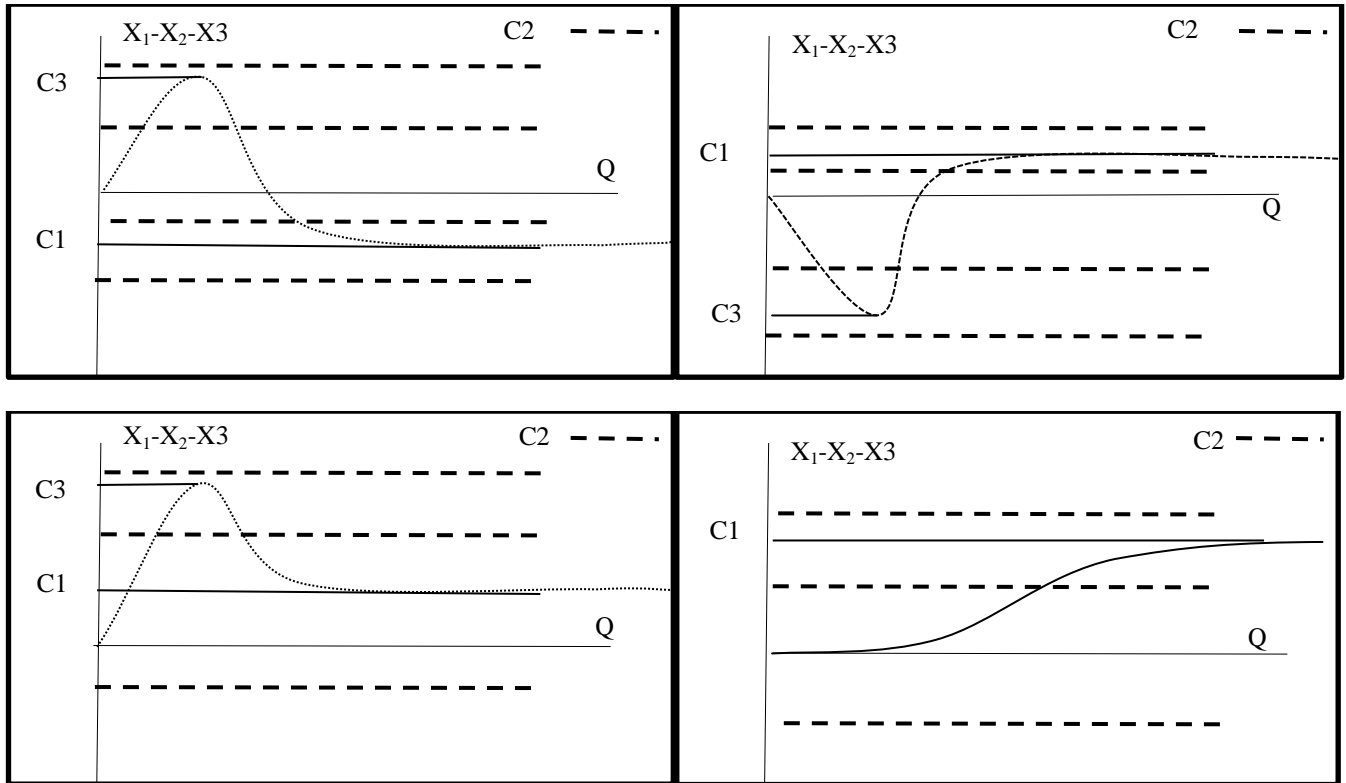
$$\lim_{Q \rightarrow +\infty} : X_1(Q) - X_2(Q) - X_3(Q) = C1$$

Equation (1) might have zero, one or two solutions depending on the values taken by the constants $C1$, $C2$ and $C3$. The following figures represent the behaviors of $X_1(Q) - X_2(Q) - X_3(Q)$

After studying these three functions, $X_1(Q) - X_2(Q) - X_3(Q)$ can have almost six behaviors, and the number of solutions will depend on the number of intersection between $C2$ and $X_1(Q) - X_2(Q) - X_3(Q)$

Here we present these six behaviors:





So as we can see the equation can have 1, 2 or 0 solutions and this number depends on the sign and order of superiority of C_1 , C_2 and C_3 . And in order to choose the right solution, we have to find the sign of $\lim_{Q \rightarrow 0} \pi(Q)$.

The following table states the optimal ordering quantity that should be considered for each case

		Number of solutions	Which solution
$C_1 \leq 0$	$C_1 \leq C_2 \leq 0$	1	Single solution
$C_1 \leq 0$	$C_3 \leq C_2 \leq C_1$	2	Lowest solution
$C_1 \leq 0$	$0 \leq C_2 \leq C_3$	2	Highest solution
$C_1 \geq 0$	$C_1 \leq C_2 \leq C_3$	2	Highest solution
$C_1 \geq 0$	$C_3 \leq C_2 \leq 0$	2	Lowest solution
All the other cases		0	No solution