Bruhat-Tits theory from Berkovich's point of view. II. Satake compactifications

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BRUHAT-TITS THEORY FROM BERKOVICH'S POINT OF VIEW. II. SATAKE COMPACTIFICATIONS OF BUILDINGS

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Abstract: In the paper *Bruhat-Tits theory from Berkovich's point of view. I — Realizations and compactifications of buildings*, we investigated various realizations of the Bruhat-Tits building $\mathscr{B}(\mathrm{G},k)$ of a connected and reductive linear algebraic group G over a non-Archimedean field k in the framework of V. Berkovich's non-Archimedean analytic geometry. We studied in detail the compactifications of the building which naturally arise from this point of view. In the present paper, we give a representation theoretic flavor to these compactifications, following Satake's original constructions for Riemannian symmetric spaces.

We first prove that Berkovich compactifications of a building coincide with the compactifications, previously introduced by the third named author and obtained by a gluing procedure. Then we show how to recover them from an absolutely irreducible linear representation of G by embedding $\mathscr{B}(\mathrm{G},k)$ in the building of the general linear group of the representation space, compactified in a suitable way. Existence of such an embedding is a special case of Landvogt's general results on functoriality of buildings, but we also give another natural construction of an equivariant embedding, which relies decisively on Berkovich geometry.

Keywords: algebraic group, local field, Berkovich geometry, Bruhat-Tits building, compactification.

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INTRODUCTION

1. Let k be field a endowed with a complete non-Archimedean absolute value, which we assume to be non-trivial. Let G be a connected reductive linear algebraic group over k. Under some assumptions on G or on k, the Bruhat-Tits building $\mathscr{B}(\mathrm{G},\mathrm{K})$ of $\mathrm{G}(\mathrm{K})$ exists for any non-Archimedean field K extending k and behaves functorially with respect to K; this is for example the case if G is quasi-split, or if k is discretely valued with a perfect residue field (in particular, if k is a local field); we refer to [**RTW09**, 1.3.4] for a discussion. Starting from this functorial existence of the Bruhat-Tits building of G over any non-Archimedean extension of k and elaborating on some results of Berkovich [**Ber90**, Chapter 5], we explained in [**RTW09**] how to realize canonically the building $\mathscr{B}(\mathrm{G},k)$ of $\mathrm{G}(k)$ in some suitable k-analytic spaces. The fundamental construction gives a canonical map from the building to the analytification G^{an} of the algebraic group G, from which one easily deduce another map from $\mathscr{B}(\mathrm{G},k)$ to X^{an}, where X stands for any generalized flag variety of G, i.e., a connected component of the projective k-scheme $\mathrm{Par}(\mathrm{G})$ parametrizing the parabolic subgroups of G. Recall that, if such a connected component X contains a k-rational point $\mathrm{P} \in \mathrm{Par}(\mathrm{G})(k)$, then X is isomorphic to the quotient scheme G/P. In more elementary words, this simply means that $\mathscr{B}(\mathrm{G},k)$ has a natural description in terms of multiplicative seminorms (of homothety classes of multiplicative seminorms, respectively) on the coordinate ring of G (on the homogeneous coordinate ring of any connected component of $\mathrm{Par}(\mathrm{G})$, respectively).

Since the algebraic scheme $\mathrm{Par}(\mathrm{G})$ is projective, the topological space underlying the analytification $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ of any connected component $\mathrm{Par}_t(\mathrm{G})$ of $\mathrm{Par}(\mathrm{G})$ is compact (that is, Hausdorff and quasi-compact), hence can be used to compactify $\mathscr{B}(\mathrm{G},k)$ by passing to the closure (in a suitable sense if k is not locally compact). In this way, one associates with each connected component $\mathrm{Par}_t(\mathrm{G})$ of $\mathrm{Par}(\mathrm{G})$ a compactified building $\overline{\mathscr{B}}_t(\mathrm{G},k)$, which is a $\mathrm{G}(k)$-topological space containing some factor of $\mathscr{B}(\mathrm{G},k)$ as a dense open subset. There is no loss of generality in restricting to connected components of $\mathrm{Par}(\mathrm{G})$ having a k-rational point, i.e., which are isomorphic to G/P for some parabolic subgroup P of G (well-defined up to $\mathrm{G}(k)$-conjugacy). Strictly speaking, $\overline{\mathscr{B}}_t(\mathrm{G},k)$ is a compactification of $\mathscr{B}(\mathrm{G},k)$ only if k is a local field and if the conjugacy class of parabolic subgroups corresponding to the component $\mathrm{Par}_t(\mathrm{G})$ of $\mathrm{Par}(\mathrm{G})$ is *non-degenerate*, i.e., consists of parabolic subgroups which do not contain a full almost simple factor of G; however, we still refer to this enlargement of $\mathscr{B}(\mathrm{G},k)$ as a "compactification" even if these conditions are not fulfilled. The compactified building $\overline{\mathscr{B}}_t(\mathrm{G},k)$ comes with a canonical stratification into locally closed subspaces indexed by a certain set of parabolic subgroups of G. The stratum attached to a parabolic subgroup P is isomorphic to the building of the semi-simplification $\mathrm{P}/\mathrm{rad}(\mathrm{P})$ of P, or rather to some factors of it. We obtain in this way one compactified building for each $\mathrm{G}(k)$-conjugacy class of parabolic subgroups of G.

2. Assuming that k is a local field, the third named author had already defined a compactification of $\mathscr{B}(\mathrm{G},k)$ for each conjugacy class of parabolic subgroup of G, see [**Wer07**]. Inspired by Satake's approach for Riemannian symmetric spaces, the construction in [loc.cit] starts with an absolutely irreducible (faithful) linear representation ρ of G and consists of two steps:

(i) the apartment $\mathrm{A}(\mathrm{S},k)$ of a maximal split torus S of G in $\mathscr{B}(\mathrm{G},k)$ is compactified, say into $\overline{\mathrm{A}}(\mathrm{S},k)_\rho$, by using the same combinatorial analysis of the weights of ρ as in [**Sat60**];

(ii) the compactified building $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$ is defined as the quotient of $\mathrm{G}(k) \times \overline{\mathrm{A}}(\mathrm{S},k)_\rho$ by a suitable extension of the equivalence relation used by Bruhat and Tits to construct $\mathscr{B}(\mathrm{G},k)$ as a quotient of $\mathrm{G}(k) \times \mathrm{A}(\mathrm{S},k)$.

It is proved in [loc.cit] that the so-obtained compactified building only depends on the position of a highest weight of ρ with respect to Weyl chambers, or equivalently on the conjugacy class of parabolic subgroups of G stabilizing the line spanned by a vector of highest weight. As suggested in [loc.cit], these compactifications turn out to coincide with Berkovich ones.

Let us define the *type* $t(\rho)$ of an absolutely irreducible linear representation $\rho : \mathrm{G} \to \mathrm{GL_V}$ as follows. If G is split, then each Borel subgroup B of G stabilizes a unique line $\mathrm{L_B}$ in V, its *highest weight line*. One easily shows that there exists a largest parabolic subgroup P of G stabilizing the line $\mathrm{L_B}$. Now, the type $t(\rho)$ of the representation ρ is characterized by the following condition: for any finite extension k'/k splitting G, the connected component $\mathrm{Par}_{t(\rho)}(\mathrm{G})$ of $\mathrm{Par}(\mathrm{G})$ contains each k'-point occurring as the largest parabolic subgroup of $\mathrm{G}\otimes_k k'$ stabilizing a highest weight line in $\mathrm{V}\otimes_k k'$. Finally, the *cotype* of the representation ρ is defined as the type of the contragredient representation $\check{\rho}$. We establish in Section 2, Theorem 2.1, the following comparison.

Theorem 1 *— Let ρ be an absolutely irreducible (faithful) linear representation of* G *in some finite-dimensional vector space over k. Then the compactifications $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$ and $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$ of the building $\mathscr{B}(\mathrm{G},k)$ are canonically isomorphic.*

3. We still assume that k is a local field but the results below hold more generally for a discretely valued non-Archimedean field with perfect residue field. Another way to compactify buildings by means of linear representations consists first in compactifying the building of the projective linear group $\mathrm{PGL_V}$ of the representation space and then using a representation in order to embed $\mathscr{B}(\mathrm{G},k)$ into this compactified building. Finally, a compactification of $\mathscr{B}(\mathrm{PGL_V},k)$ can be obtained by embedding this building in some projective space, hence this viewpoint is the closest one in spirit to the original approach for symmetric spaces. It is also a way to connect Bruhat-Tits theory to Berkovich's interpretation of the space of seminorms on a given k-vector space [**Ber95**].

More precisely, let $\rho : \mathrm{G} \to \mathrm{GL_V}$ be an absolutely irreducible linear representation of G in a finite-dimensional k-vector space V. We use such a map ρ in two ways to obtain continuous $\mathrm{G}(k)$-equivariant maps from the building $\mathscr{B}(\mathrm{G},k)$ to a compact space $\mathscr{X}(\mathrm{V},k)$ naturally attached to the k-vector space V. Denoting by $\mathscr{S}(\mathrm{V},k)$ the "extended Goldman-Iwahori space" consisting of non-zero seminorms on V (the space of norms was studied in [**GI63**]), then the space $\mathscr{X}(\mathrm{V},k)$ is the quotient of $\mathscr{S}(\mathrm{V},k)$ by homotheties. It is the non-Archimedean analogue of the quotient of the cone of positive (possibly degenerate) Hermitian matrices in the projective space associated with $\mathrm{End}(\mathrm{V})$ [**Sat60**].

In the real case, the latter space is classically the target space of a suitable Satake map. In our case, we identify $\mathscr{X}(\mathrm{V},k)$ with the compactification $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ corresponding to the type δ of parabolic subgroups stabilizing a *hyperplane* of V. One could also consider the compactified building $\overline{\mathscr{B}}_\pi(\mathrm{PGL_V},k)$ associated with the type π of parabolic subgroups stabilizing a *line* of V (see [**Wer01**]). Note that $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k) \cong \overline{\mathscr{B}}_\pi(\mathrm{PGL_{V^\vee}},k)$, where $\mathrm{V}^\vee$ is the dual of V.

A first way to obtain a map $\mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$ is to make use of E. Landvogt's work on the functoriality of Bruhat-Tits buildings (with respect both to the group and to the field). Indeed, specializing the results of [**Lan00**] to k-homomorphisms arising from linear representations $\rho : \mathrm{G} \to \mathrm{GL_V}$, we obtain a (possibly non-uniquely defined) map $\rho_* : \mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{PGL_V},k)$ between buildings. We can then compose it with the compactification map $\vartheta_\pi : \mathscr{B}(\mathrm{PGL_V},k) \to \overline{\mathscr{B}}_\pi(\mathrm{PGL_V},k)$ in order to obtain an analogue of a Satake map.

There is another way to embed the building $\mathscr{B}(\mathrm{G},k)$ into $\mathscr{X}(\mathrm{V},k)$, which turns out to be very natural and relies crucially on Berkovich geometry. There exists a natural k-morphism $\widetilde{\rho}$ from the scheme $\mathrm{Bor}(\mathrm{G})$ of Borel subgroups of G to the projective space $\mathbb{P}(\mathrm{V})$ satisfying the following condition: for any extension K/k, the map $\widetilde{\rho}_\mathrm{K}$ sends a Borel subgroup B of $\mathrm{G}\otimes_k \mathrm{K}$ to the unique K-point $\widetilde{\rho}(\mathrm{B})$ of $\mathbb{P}(\mathrm{V})$ it fixes. By passing to analytic spaces, we get a map $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G})^{\mathrm{an}} \to \mathbb{P}(\mathrm{V})^{\mathrm{an}}$. Using the concrete description of $\mathscr{X}(\mathrm{V},k)$ and $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$, we have a natural retraction $\tau : \mathbb{P}(\mathrm{V})^{\mathrm{an}} \to \mathscr{X}(\mathrm{V},k)$, so that the composition $\underline{\rho} = \tau \circ \widetilde{\rho} \circ \vartheta_\varnothing$ sends the Bruhat-Tits building $\mathscr{B}(\mathrm{G},k)$ into $\mathscr{X}(\mathrm{V},k)$. This is our second way to obtain a non-Archimedean analogue of a Satake map, and it is easily seen that this canonical map sends an apartment into an apartment.

These two embedding procedures lead to the previous families of compactifications (cf. Theorem 4.8 and Theorem 5.3):

Theorem 2 *— Assume that k is a non-Archimedean local field and let $\rho : \mathrm{G} \to \mathrm{GL}_{\mathrm{V}}$ be an absolutely irreducible linear representation of* G *in a finite-dimensional vector space* V *over k.*

(i) *The map $\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$ induces a $\mathrm{G}(k)$-equivariant homeomorphism between $\overline{\mathscr{B}}_{t(\check{\rho})}(\mathrm{G},k)$ and the closure of the image of $\underline{\rho}$ in $\mathscr{X}(\mathrm{V},k)$.*

(ii) *Any Landvogt map $\rho_* : \mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$ induces a $\mathrm{G}(k)$-equivariant homeomorphism between $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$ and the closure of its image in $\overline{\mathscr{B}}_{\pi}(\mathrm{PGL}_{\mathrm{V}},k)$.*

Conventions. Assumptions on the field *k* are made explicit at the beginning of each section. Notations and conventions from [**RTW09**] are recalled in section 1.

Let us stress one particular working hypothesis: the results in **[loc.cit]** were obtained under a functoriality assumption for buildings with respect to non-Archimedean extension of the ground field (see **[loc.cit, 1.3.4]** for a precise formulation). This assumption, which is fulfiled in particular if *k* is discretely valued with perfect residue field or if the group under consideration is split, is made throughout the present work.

Structure of the paper. In the first section, we briefly review the constructions of [**RTW09**] and state the results from [loc.cit] to be used in this work. The second section is devoted to the identification of Berkovich compactifications with the compactifications introduced in [**Wer07**]. The third section contains a concrete description of the Berkovich compactification of the building $\mathscr{X}(\mathrm{V},k) = \mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$ associated with the projective space $\mathbb{P}(\mathrm{V})$ seen as a generalized flag variety. The last two sections deal with the recovery of Berkovich compactifications via embeddings into $\mathscr{X}(\mathrm{V},k)$, in the spirit of Satake's original construction for Riemannian symmetric spaces. In Section 4, we construct a canonical $\mathrm{G}(k)$-map from $\mathscr{B}(\mathrm{G},k)$ to $\mathscr{X}(\mathrm{V},k)$ for each absolutely irreducible linear representation of G in V, and we show that taking the closure leads to the Berkovich compactification of $\mathscr{B}(\mathrm{G},k)$ of type $t(\check{\rho})$. In Section 5, we rely on Landvogt's functoriality results to produce such a map and derive the same conclusion.

1. BERKOVICH COMPACTIFICATIONS OF BUILDINGS

This section provides a brief summary of realizations and compactifications of Bruhat-Tits buildings in the framework of Berkovich's non-Archimedean analytic geometry. We refer to [**RTW09**] for proofs, details and complements.

In the following, we consider a non-Archimedean field k, i.e., a field endowed with a complete non-Archimedean absolute value which we assume to be non-trivial, and a semisimple and connected linear k-group G.

(1.1) For each point x of the Bruhat-Tits building $\mathscr{B}(\mathrm{G},k)$, there exists a unique affinoid subgroup G_x of G^{an} satisfying the following condition: for any non-Archimedean extension K/k, the group $\mathrm{G}_x(\mathrm{K})$ is the stabilizer of x_{K} in $\mathrm{G}(\mathrm{K})$, where x_{K} denotes the image of x under the natural injection $\mathscr{B}(\mathrm{G},k) \hookrightarrow \mathscr{B}(\mathrm{G},\mathrm{K})$. Seen as a set of multiplicative seminorms on the coordinate algebra $\mathscr{O}(\mathrm{G})$ of G, the subspace G_x contains a unique maximal point, denoted by $\vartheta(x)$. One can recover G_x from $\vartheta(x)$ as its holomorphic envelope:

$$\mathrm{G}_x = \{z \in \mathrm{G}^{\mathrm{an}} \,;\, |f|(z) \leqslant |f|(\vartheta(x)) \ \text{ for all } f \in \mathscr{O}(\mathrm{G})\}.$$

We have thus defined a map

$$\vartheta : \mathscr{B}(\mathrm{G},k) \to \mathrm{G}^{\mathrm{an}}$$

which is continuous, injective and $\mathrm{G}(k)$-equivariant with respect to the $\mathrm{G}(k)$-action by conjugation on G^{an}. By its very construction ϑ is compatible with non-Archimedean extensions of k.

(1.2) We let $\mathrm{Par}(\mathrm{G})$ denote the k-scheme of parabolic subgroups of G; this is a smooth and projective scheme representing the functor

$$\mathbf{Sch}/k \to \mathbf{Sets}, \ \ \mathrm{S} \mapsto \{\text{parabolic subgroups of } \mathrm{G} \times_k \mathrm{S}\}.$$

The connected components of $\mathrm{Par}(\mathrm{G})$ are naturally in bijection with $\mathrm{Gal}(k^a|k)$-stable subsets of vertices in the Dynkin diagram of $\mathrm{G} \otimes_k k^a$. Such a subset t is called a *type* of parabolic subgroups of G and we denote by $\mathrm{Par}_t(\mathrm{G})$ the corresponding connected component of $\mathrm{Par}(\mathrm{G})$. For example, $\mathrm{Par}_\varnothing(\mathrm{G})$ is the scheme of Borel subgroups of G whereas the trivial type corresponds to the maximal parabolic subgroup G. Finally, a type t is said to be *k-rational* if $\mathrm{Par}_t(\mathrm{G})(k) \neq \varnothing$, i.e., if there exists a parabolic subgroup of G of type t.

With each parabolic subgroup P of G is associated a morphism $\omega_{\mathrm{P}} : \mathrm{G} \to \mathrm{Par}(\mathrm{G})$, defined functor-theoretically by $g \mapsto g\mathrm{P}g^{-1}$ and inducing an isomorphism from G/P to the (geometrically) connected component of $\mathrm{Par}(\mathrm{G})$ containing the k-point P. Composing ϑ with the analytification of ω_{P}, we obtain a continuous and $\mathrm{G}(k)$-equivariant map from $\mathscr{B}(\mathrm{G},k)$ to $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$ which depends only on the type t of P. This map is denoted by ϑ_t and its image lies in the connected component $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ of $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$. The map ϑ_t only depends on the type t, not on the choice of P in $\mathrm{Par}_t(\mathrm{G})(k)$. It is defined more generally for any type t of parabolic subgroups, even non-k-rational ones; however, we restrict to k-rational types in this section.

The topological space underlying $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$ is compact, hence leads to compactifications of the building $\mathscr{B}(\mathrm{G},k)$ by closing. From now on, we fix a k-rational type t and describe the corresponding compactification of $\mathscr{B}(\mathrm{G},k)$. If S is a maximal split torus of G, we recall that $\mathrm{A}(\mathrm{S},k)$ denotes the corresponding apartment in the building $\mathscr{B}(\mathrm{G},k)$.

Definition 1.1. — *For any maximal split torus* S *of* G*, we let* $\overline{\mathrm{A}}_t(\mathrm{S},k)$ *denote the closure of* $\vartheta_t(\mathrm{A}(\mathrm{S},k))$ *in* $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$*. We set*

$$\overline{\mathscr{B}}_t(\mathrm{G},k) = \bigcup_{\mathrm{S}} \overline{\mathrm{A}}_t(\mathrm{S},k) \subset \mathrm{Par}(\mathrm{G})^{\mathrm{an}},$$

where the union is taken over the set of maximal split tori of G*. This is a* $\mathrm{G}(k)$*-invariant subset of* $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$*, which we endow with the quotient topology induced by the natural* $\mathrm{G}(k)$*-equivariant map*

$$\mathrm{G}(k) \times \overline{\mathrm{A}}_t(\mathrm{S},k) \to \overline{\mathscr{B}}_t(\mathrm{G},k).$$

(See [**RTW09**, Definition 3.30].)

The type t is said to be *non-degenerate* if it restricts non-trivially to each almost simple factor of G, i.e., if t, seen as a $\mathrm{Gal}(k^a|k)$-stable set of vertices in the Dynkin diagram D of $\mathrm{G} \otimes_k k^a$, does not contain any connected component of D. In general, there exist two semisimple groups H', H'' and a central isogeny $\mathrm{G} \to \mathrm{H}' \times \mathrm{H}''$ such that t has non-degenerate restriction to H' and trivial restriction to H''. In this situation, $\mathscr{B}(\mathrm{G},k) \cong \mathscr{B}(\mathrm{H}',k) \times \mathscr{B}(\mathrm{H}'',k)$ and we let $\mathscr{B}_t(\mathrm{G},k)$ denote the factor $\mathscr{B}(\mathrm{H}',k)$.

Proposition 1.2. — (i) *The map* $\vartheta_t : \mathscr{B}(\mathrm{G},k) \to \mathrm{Par}(\mathrm{G})^{\mathrm{an}}$ *factors through the canonical projection of* $\mathscr{B}(\mathrm{G},k)$ *onto* $\mathscr{B}_t(\mathrm{G},k)$ *and induces an injection of the latter building in* $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$.

(ii) *If the field* k *is locally compact, then* $\overline{\mathscr{B}}_t(\mathrm{G},k)$ *is the closure of* $\vartheta_t(\mathscr{B}(\mathrm{G},k))$ *in* $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$*, endowed with the induced topology.*

(See [**RTW09**, Proposition 3.34].)

If k is not locally compact, the topological space $\overline{\mathscr{B}}_t(\mathrm{G},k)$ is not compact. However, the map $\vartheta_t : \mathscr{B}_t(\mathrm{G},k) \hookrightarrow \overline{\mathscr{B}}_t(\mathrm{G},k)$ still induces a homeomorphism onto an open dense subset of $\overline{\mathscr{B}}_t(\mathrm{G},k)$.

(1.3) The topological space $\overline{\mathscr{B}}_t(\mathrm{G},k)$ carries a canonical stratification whose strata are lower-dimensional buildings coming from semisimplications of suitable parabolic subgroups of G.

We can attach to each parabolic subgroup Q of G a closed and smooth subscheme $\mathrm{Osc}_t(\mathrm{Q})$ of $\mathrm{Par}_t(\mathrm{G})$, homogeneous under Q and representing the subfunctor

$$\mathbf{Sch}/k \to \mathbf{Sets}, \ \ \mathrm{S} \mapsto \left\{ \begin{array}{c} \text{parabolic subgroups of } \mathrm{G} \times_k \mathrm{S} \\ \text{of type } t\text{, osculatory with } \mathrm{Q} \times_k \mathrm{S} \end{array} \right\}.$$

We recall that two parabolic subgroups of a reductive S-group scheme are osculatory if, étale locally on S, they contain a common Borel subgroup. Letting Q_{ss} denote the semisimple k-group $\mathrm{Q}/\mathrm{rad}(\mathrm{Q})$, the morphism $\iota_{\mathrm{Q}} : \mathrm{Osc}_t(\mathrm{Q}) \to \mathrm{Par}_t(\mathrm{Q}_{\mathrm{ss}})$ defined functor-theoretically by $\mathrm{P} \mapsto (\mathrm{P} \cap \mathrm{Q})/\mathrm{rad}(\mathrm{Q})$ is an isomorphism.

There exists a largest parabolic subgroup Q' stabilizing $\mathrm{Osc}_t(\mathrm{Q})$. By construction, we have $\mathrm{Q} \subset \mathrm{Q}'$ and $\mathrm{Osc}_t(\mathrm{Q}') = \mathrm{Osc}_t(\mathrm{Q})$, and we say that Q is *$t$-relevant* if $\mathrm{Q} = \mathrm{Q}'$. In general, Q' is the smallest t-relevant parabolic subgroup of G containing Q.

Example 1.3. — a) It $t_{\min}$ denotes the type of minimal parabolic subgroups of G, then each parabolic subgroup of G is $t_{\min}$-relevant. Indeed, for any two parabolic subgroups P and Q such that $\mathrm{Q} \subsetneq \mathrm{P}$, there exists a minimal parabolic subgroup contained in P but not in Q; this implies $\mathrm{Osc}_{t_{\min}}(\mathrm{Q}) \neq \mathrm{Osc}_{t_{\min}}(\mathrm{P})$, hence Q is the largest parabolic subgroup stabilizing $\mathrm{Osc}_{t_{\min}}(\mathrm{Q})$.

b) Let V be a finite-dimensional k-vector space. We assume that $\mathrm{G} = \mathrm{PGL}_{\mathrm{V}}$ and that δ is the type of parabolic subgroups of $\mathrm{PGL}_{\mathrm{V}}$ stabilizing a hyperplane. In this case, $\mathrm{Par}_\delta(\mathrm{G})$ is the projective space $\mathbb{P}(\mathrm{V})$, i.e., the scheme of hyperplanes in V. Each parabolic subgroup Q of $\mathrm{PGL}_{\mathrm{V}}$ is the stabilizer of a well-defined flag $\mathrm{V}^\bullet$ of linear subspaces, and two parabolic subgroups are osculatory if and only if the corresponding flags admit a common refinement, i.e., are subflags of the same flag. It follows that $\mathrm{Osc}_\delta(\mathrm{Q})$ is the closed subscheme $\mathbb{P}(\mathrm{V}/\mathrm{W})$ of $\mathbb{P}(\mathrm{V})$, where W is the largest proper linear subspace of V occurring in the flag $\mathrm{V}^\bullet$, and therefore δ-relevant parabolic subgroups of $\mathrm{PGL}_{\mathrm{V}}$ are precisely the stabilizers of flags $(\{0\} \subset \mathrm{W} \subset \mathrm{V})$, where W is any linear subspace of V.

We can now describe the canonical stratification on the compactified building $\overline{\mathscr{B}}_t(\mathrm{G},k)$.

Theorem 1.4. — *For any parabolic subgroup* Q *of* G*, we use the map* $\iota_{\mathrm{Q}}^{-1} \circ \vartheta_t$ *to embed* $\mathscr{B}_t(\mathrm{Q}_{\mathrm{ss}},k)$ *into* $\mathrm{Osc}_t(\mathrm{Q})^{\mathrm{an}} \subset \mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$.

(i) *As a subset of* $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$*, the building* $\mathscr{B}(\mathrm{Q}_{\mathrm{ss}},k)$ *is contained in* $\overline{\mathscr{B}}_t(\mathrm{G},k)$.

(ii) *We have the following stratification by locally closed subsets:*

$$\overline{\mathscr{B}}_t(\mathrm{G},k) = \bigsqcup_{t\text{-relevant } Q'\mathrm{s}} \mathscr{B}_t(\mathrm{Q}_{\mathrm{ss}},k),$$

where the union is indexed by the t-relevant parabolic subgroups of G. *The closure of the stratum* $\mathscr{B}_t(\mathrm{Q}_{\mathrm{ss}},k)$ *is the union of all strata* $\mathscr{B}_t(\mathrm{P}_{\mathrm{ss}},k)$ *with* $\mathrm{P}\subset\mathrm{Q}$ *and is canonically homeomorphic to the compactified building* $\overline{\mathscr{B}}_t(\mathrm{Q}_{\mathrm{ss}},k)$.

(See [**RTW09**, Theorem 4.1].)

Example 1.5. — a) Suppose that $t = t_{\min}$ is the type of minimal parabolic subgroups of G. This type is non-degenerate and each parabolic subgroup of G is $t_{\min}$-relevant, hence the boundary of $\overline{\mathscr{B}}_{t_{\min}}(\mathrm{G},k)$ contains a copy of the building of Q_{ss} for each proper parabolic subgroup Q of G.

b) Let V be a finite-dimensional k-vector space. We assume that $\mathrm{G} = \mathrm{PGL}_{\mathrm{V}}$ and that $t = \delta$ is the type of parabolic subgroups of $\mathrm{PGL}_{\mathrm{V}}$ stabilizing a hyperplane. In this case, the boundary of $\overline{\mathscr{B}}_\delta(\mathrm{PGL}_{\mathrm{V}},k)$ is the union of the buildings $\mathscr{B}(\mathrm{PGL}(\mathrm{V}/\mathrm{W}),k)$, where W runs over the set of proper non-zero linear subspaces of V.

(1.4) We now look at the compactified apartment $\overline{\mathrm{A}}_t(\mathrm{S},k)$ of a maximal split torus S of G. The apartment $\mathrm{A}(\mathrm{S},k)$ is an affine space under the vector space $\mathrm{V}(\mathrm{S}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(\mathrm{S}),\mathbb{R})$, where $\mathrm{X}^*(\mathrm{S}) = \mathrm{Hom}_{k-\mathbf{Gr}}(\mathrm{S},\mathbb{G}\mathrm{m}_k)$ is the group of characters of S. Let $\Phi = \Phi(\mathrm{G},\mathrm{S}) \subset \mathrm{X}^*(\mathrm{S})$ denote the set of roots of G with respect to S. With each parabolic subgroup P of G containing S we associate its *Weyl cone*

$$\mathfrak{C}(\mathrm{P}) = \{u \in \mathrm{V}(\mathrm{S})\ ;\ \langle\alpha,u\rangle \geqslant 0 \ \text{ for all roots } \alpha \text{ of P}\},$$

which is a strictly convex rational polyhedral cone in $\mathrm{V}(\mathrm{S})$. The collection of Weyl cones of parabolic subgroups of G containing S is a complete *fan* on the vector space $\mathrm{V}(\mathrm{S})$, i.e., a finite family of strictly convex rational polyhedral cones stable under intersection, in which any two cones intersect along a common face, and satisfying the additional condition that $\mathrm{V}(\mathrm{S})$ is covered by the union of these cones.

Relying on the k-rational type t, we can define a new complete fan on $\mathrm{V}(\mathrm{S})$, which we denote by $\mathscr{F}_t$. The fan of Weyl cones will turn out to be $\mathscr{F}_{t_{\min}}$. First of all, if P is a parabolic subgroup of type t containing S, we define $\mathrm{C}_t(\mathrm{P})$ as the "combinatorial neighborhood" of $\mathfrak{C}(\mathrm{P})$ in $\mathrm{V}(\mathrm{S})$, i.e.,

$$\mathrm{C}_t(\mathrm{P}) = \bigcup_{\substack{\mathrm{Q\ parabolic}\\ \mathrm{S}\subset\mathrm{Q}\subset\mathrm{P}}} \mathfrak{C}(\mathrm{Q}).$$

This is a convex rational polyhedral cone, and $\mathrm{C}_t(\mathrm{P})$ is strictly convex if and only if the type t is non-degenerate. More precisely, the central isogeny $\mathrm{G} \to \mathrm{H}' \times \mathrm{H}''$ introduced after Definition 1.1 corresponds to a decomposition of Φ as the union $\Phi' \cup \Phi''$ of two closed and disjoint subsets, and the largest linear subspace of $\mathrm{C}_t(\mathrm{P})$ is the vanishing locus of Φ'', namely

$$\langle\Phi''\rangle = \{u \in \mathrm{X}^*(\mathrm{S})\ ;\ \langle\alpha,u\rangle = 0 \ \text{ for all } \alpha \in \Phi''\}.$$

When P runs over the set of parabolic subgroups of G of type t and containing S, one checks that the set $\mathscr{F}_t$, consisting of the cones $\mathrm{C}_t(\mathrm{P})$ together with their faces, induces a complete fan on the quotient space $\mathrm{V}(\mathrm{S})/\langle\Phi''\rangle$.

Any strictly convex rational polyhedral cone C in $\mathrm{V}(\mathrm{S})$ has a canonical compactification $\overline{\mathrm{C}}$, whose description is nicer if we switch to multiplicative notation for the real dual of $\mathrm{X}^*(\mathrm{S})$. Hence, we set $\Lambda(\mathrm{S}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(\mathrm{S}),\mathbb{R}_{>0})$ and use the isomorphism $\mathbb{R} \to \mathbb{R}_{>0},\ x \mapsto e^x$ in order to identify $\mathrm{V}(\mathrm{S})$ with $\Lambda(\mathrm{S})$.

Let M denote the set of characters $\chi \in \mathrm{X}^*(\mathrm{S})$ such that $\langle\chi,u\rangle \leqslant 1$ for any $u \in \mathrm{C} \subset \Lambda(\mathrm{S})$. This is a finitely generated semigroup of $\mathrm{X}^*(\mathrm{S})$ and the map

$$\mathrm{C} \to \mathrm{Hom}_{\mathbf{Mon}}(\mathrm{M},]0,1]),\ \ u \mapsto (\chi \mapsto \langle\chi,u\rangle)$$

identifies C with the set $\mathrm{Hom}_{\mathbf{Mon}}(\mathrm{M},]0,1])$ of morphisms of unitary monoids, endowed with the coarsest topology making each evaluation map continuous. We define $\overline{\mathrm{C}}$ as the set $\mathrm{Hom}_{\mathbf{Mon}}(\mathrm{M},[0,1])$ endowed with the analogous topology; this is a compact space in which C embeds as an open dense subspace. Each complete fan $\mathscr{F}$ of strictly convex rational polyhedral cones on $\Lambda(\mathrm{S})$ gives rise to a compactification $\overline{\Lambda}(\mathrm{S})^{\mathscr{F}}$ of this vector space, defined by gluing together the compactifications of the cones $\mathrm{C}\in\mathscr{F}$. More generally, one can compactify in this way any affine space under $\Lambda(\mathrm{S})$.

Proposition 1.6. — *Let* S *be a maximal split torus of* G. *The compactified apartment* $\overline{\mathrm{A}}_t(\mathrm{S},k)$ *is canonically homeomorphic to the compactification of* $\mathrm{A}(\mathrm{S},k)/\langle\Phi''\rangle$ *associated with the complete fan* $\mathscr{F}_t$.

(See [**RTW09**, Proposition 3.35].)

The connection between t-relevant parabolic subgroups on the one hand and cones belonging to $\mathscr{F}_t$ on the other hand is the following.

Proposition 1.7. — *For each parabolic subgroup* Q *of* G *containing* S, *there is a smallest cone* $\mathrm{C}_t(\mathrm{Q})$ *in* $\mathscr{F}_t$ *containing the Weyl cone* $\mathfrak{C}(\mathrm{Q})$. *The following two conditions are equivalent:*

(i) Q *is t-relevant;*
(ii) Q *is the largest parabolic subgroup defining the cone* $\mathrm{C}_t(\mathrm{Q})$.

In particular, the map $\mathrm{Q}\mapsto\mathrm{C}_t(\mathrm{Q})$ *gives a one-to-one correspondence between t-relevant parabolic subgroups containing* S *and cones in the fan* $\mathscr{F}_t$.

(See [**RTW09**, Remark 3.25].)

(1.5) For any parabolic subgroup Q of G containing S, the cone $\mathrm{C}_t(\mathrm{Q})$ admits the following root-theoretic description. Let P be a parabolic subgroup of type t osculatory with Q. We have

$$\mathrm{C}_t(\mathrm{P})=\{z\in\Lambda(\mathrm{S})\ ;\ \langle\alpha,z\rangle\leqslant 1 \text{ for all } \alpha\in\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{P}^{\mathrm{op}}),\mathrm{S})\},$$

and $\mathrm{C}_t(\mathrm{Q})$ is the face of $\mathrm{C}_t(\mathrm{P})$ cut out by the linear subspace

$$\langle\mathrm{C}_t(\mathrm{Q})\rangle=\{z\in\Lambda(\mathrm{S})\ ;\ \langle\alpha,z\rangle=1 \text{ for all } \alpha\in\Phi(\mathrm{L}_{\mathrm{Q}},\mathrm{S})\cap\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{P}^{\mathrm{op}}),\mathrm{S})\},$$

where $\mathrm{rad}^{\mathrm{u}}(\cdot)$ stands for the unipotent radical and L_{Q} denotes the Levi subgroup of Q associated with S ([**RTW09**, Lemma 3.15]).

One deduces the following root-theoretical characterization of t-relevancy. Let S be a maximal split torus of G. We fix a minimal parabolic subgroup P_0 of G containing S and write Δ for the corresponding basis of $\Phi(\mathrm{G},\mathrm{S})$, which we identify with the set of vertices in the Dynkin diagram of G. The map

$$\left\{\begin{array}{c}\text{parabolic subgroups of G}\\ \text{containing S}\end{array}\right\}\to\{\text{subsets of }\Delta\},\quad \mathrm{Q}\mapsto\mathrm{Y}_{\mathrm{Q}}=\Delta\cap\Phi(\mathrm{L}_{\mathrm{Q}},\mathrm{S})$$

is a bijection.

Proposition 1.8. — *Let* Q *be a parabolic subgroup of* G. *We denote by* Y_t *the subset of* Δ *associated with the parabolic subgroup of type t containing* P_0 *and let* $\widetilde{\mathrm{Y}_{\mathrm{Q}}}$ *denote the union of the connected components of* Y_{Q} *meeting* $\Delta-\mathrm{Y}_t$.

(i) *The parabolic subgroup* Q *is t-relevant if and only if for any root* $\alpha\in\Delta$, *we have*

$$(\alpha\in\mathrm{Y}_t \text{ and } \alpha\perp\widetilde{\mathrm{Y}_{\mathrm{Q}}})\Longrightarrow\alpha\in\mathrm{Y}_{\mathrm{Q}}.$$

(ii) *More generally, the smallest t-relevant parabolic subgroup of* G *containing* Q *is associated with the subset of* Δ *obtained by adjoining to* Y_{Q} *all roots in* Y_t *which are orthogonal to each connected component of* Y_{Q} *meeting* $\Delta-\mathrm{Y}_t$.

(iii) *The linear subspace of* $\Lambda(\mathrm{S})$ *spanned by the cone* $\mathrm{C}_t(\mathrm{Q})$ *is the vanishing locus of* $\widetilde{\mathrm{Y}_{\mathrm{Q}}}$:

$$\langle\mathrm{C}_t(\mathrm{Q})\rangle=\{z\in\Lambda(\mathrm{S})\ ;\ \langle\alpha,z\rangle=1 \text{ for all } \alpha\in\widetilde{\mathrm{Y}_{\mathrm{Q}}}\}.$$

(For assertions (i) and (ii), see [**RTW09**, Proposition 3.24] and [**RTW09**, Remark 3.25, 2]. Assertion (iii) follows from [**RTW09**, Proposition 3.22] and [**RTW09**, Remark 3.25, 2].)

Here, orthogonality is understood with respect to a scalar product on $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ invariant under the Weyl group of $\Phi(G,S)$.

Remark 1.9. — Given a maximal split torus S and a parabolic subgroup Q containing S, we have the following inclusions of cones

$$\mathfrak{C}(Q) = C_{\varnothing}(Q) \subset C_t(Q) \subset C_{t(Q)}(Q)$$

for any k-rational type t. Up to a central isogeny, we can write L_Q as the product $L' \times L''$ of two reductive groups such that t has non-degenerate restriction to L' and trivial restriction to L''. This amounts to decomposing $\Phi(L_Q,S)$ as the union of two disjoint closed subsets $\Phi(L',S)$ and $\Phi(L'',S)$, with

$$\Phi(L',S) = \langle \widetilde{Y_Q} \rangle \cap \Phi(G,S)$$

if we use the notation introduced in the preceding proposition. It follows from the latter that the cone $C_t(Q)$ is the intersection of $C_{t(Q)}(Q)$ with the linear subspace of $\Lambda(S)$ cut out by all roots in $\Phi(L',S)$.

(1.6) Finally, we describe the stabilizer of a point of $\overline{\mathscr{B}}_t(G,k)$.

Theorem 1.10. — *Let x be a point in $\overline{\mathscr{B}}_t(G,k)$ and let Q denote the t-relevant parabolic subgroup of G corresponding to the stratum containing x.*

1. *There exists a largest smooth and connected closed subgroup $R_t(Q)$ of G satisfying the following conditions:*
   - *$R_t(Q)$ is a normal subgroup of Q and contains $\mathrm{rad}(Q)$;*
   - *for any non-Archimedean extension K/k, the subgroup $R_t(Q)(K)$ of $G(K)$ acts trivially on the stratum $\mathscr{B}(Q_{ss},K)$.*
2. *The canonical projection $Q_{ss} \to Q/R_t(Q)$ identifies the buildings $\mathscr{B}_t(Q_{ss},k)$ and $\mathscr{B}(Q/R_t(Q),k)$.*
3. *There exists a unique geometrically reduced k-analytic subgroup $\mathrm{Stab}^t_G(x)$ of G^{an} such that, for any non-Archimedean extension K/k, the group $\mathrm{Stab}^t_G(x)(K)$ is the subgroup of $G(K)$ fixing x in $\overline{\mathscr{B}}_t(G,K)$.*
4. *We have $R_t(Q)^{an} \subset \mathrm{Stab}^t_G(x)^{an} \subset Q^{an}$ and the canonical isomorphism $Q^{an}/R_t(Q)^{an} \cong (Q/R_t(Q))^{an}$ identifies the quotient group $\mathrm{Stab}^t_G(x)/R_t(Q)^{an}$ with the affinoid subgroup $(Q/R_t(Q))_x$ of $(Q/R_t(Q))^{an}$ attached in (1.1) to the point x of $\mathscr{B}_t(Q_{ss},k) = \mathscr{B}(Q/R_t(Q),k)$.*

(See [**RTW09**, Proposition 4.7 and Theorem 4.11].)

Remark 1.11. — If Q is a proper t-relevant parabolic subgroup of G, then $\mathrm{rad}(Q)(k)$ is an unbounded subgroup of $G(k)$. Since $\mathrm{rad}(Q) \subset R_t(Q) \subset \mathrm{Stab}^t_G(x)$ for any $x \in \mathscr{B}_t(Q_{ss},k)$, it follows that any point lying in the boundary $\overline{\mathscr{B}}_t(G,k) - \mathscr{B}_t(G,k)$ has an unbounded stabilizer in $G(k)$. If the type t is non-degenerate, the converse assertion is true.

We can give a more precise description of the subgroup $\mathrm{Stab}^t_G(x)(k)$ of $G(k)$ stabilizing a point x of $\overline{\mathscr{B}}_t(G,k)$. Let us fix some notation. We pick a maximal split torus S of G whose compactified apartment contains x and set $N = \mathrm{Norm}_G(S)$. Let Q denote the t-relevant parabolic subgroup of G attached to the stratum containing x and write L for the Levi factor of Q with respect to S. We set $L'' = R_t(Q) \cap L$ and let L' denote the semisimple subgroup of L generated by the isotropic almost simple components of L on which t is non-trivial. Both the product morphism $L' \times L'' \to L$ and the morphism $L' \to Q/R_t(Q)$ are central isogenies. We introduce also the split tori $S' = (L' \cap S)^\circ$ and $S'' = (L'' \cap S)^\circ$.

Let $N(k)_x$ denote the stabilizer of x in the $N(k)$-action on $\overline{A}_t(S,k)$. Finally, we fix a special point in $A(S,k)$ and we recall that, for each root $\alpha \in \Phi(G,S)$, Bruhat-Tits theory endows the group $U_\alpha(k)$ with a decreasing filtration $\{U_\alpha(k)_r\}_{r \in [-\infty,\infty]}$.

Theorem 1.12. — *Let x be a point in $\overline{\mathscr{B}}_t(\mathrm{Q},k)$ and let* Q *denote the t-relevant parabolic subgroup of* G *attached to the stratum containing x.*

The group $\mathrm{Stab}^t_{\mathrm{G}}(x)(k)$ *is Zariski dense in* Q *and is generated by the following subgroups of* $\mathrm{G}(k)$*:*

- $\mathrm{N}(k)_x$*;*
- *all* $\mathrm{U}_\alpha(k)$ *with* $\alpha \in \Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}),\mathrm{S})$*;*
- *all* $\mathrm{U}_\alpha(k)$ *with* $\alpha \in \Phi(\mathrm{L}'',\mathrm{S}'')$*;*
- *all* $\mathrm{U}_\alpha(k)_{-\log\alpha(x)}$ *with* $\alpha \in \Phi(\mathrm{L}',\mathrm{S}')$

(See [**RTW09**, Theorem 4.14].)

An easy consequence of this description of stabilizers is the following generalization of well-known properties of Bruhat-Tits buildings.

Theorem 1.13. — *1. Let* S *be a maximal split torus of* G *and set* $\mathrm{N} = \mathrm{Norm}_{\mathrm{G}}(\mathrm{S})$*. The compactified building* $\overline{\mathscr{B}}_t(\mathrm{G},k)$ *is the topological quotient of* $\mathrm{G}(k) \times \overline{\mathrm{A}}_t(\mathrm{S},k)$ *by the following equivalence relation:*

$$(g,x) \sim (h,y) \Longleftrightarrow \left(\exists n \in \mathrm{N}(k),\ y = n\cdot x \ \text{ and } g^{-1}hn \in \mathrm{Stab}^t_{\mathrm{G}}(x)(k)\right).$$

2. Let x and y be two points in $\overline{\mathscr{B}}_t(\mathrm{G},k)$*.*

(i) *There exists a maximal split torus* S *in* G *such that x and y lie in* $\overline{\mathrm{A}}_t(\mathrm{S},k)$*.*
(ii) *The group* $\mathrm{Stab}^t_{\mathrm{G}}(x)(k)$ *acts transitively on the compactified apartments containing x.*
(iii) *We have the following* mixed Bruhat decomposition*:*

$$\mathrm{G}(k) = \mathrm{Stab}^t_{\mathrm{G}}(x)(k)\mathrm{N}(k)\mathrm{Stab}^t_{\mathrm{G}}(y)(k).$$

(See [**RTW09**, Corollary 4.15 and Theorem 4.20].)

(1.7) Many statements listed above are proved by using an explicit formula for the map ϑ_t when G is *split*.

Let P be a parabolic subgroup of G of type t and pick a maximal split torus S of G contained in P. The morphism

$$\mathrm{rad}^{\mathrm{u}}(\mathrm{P}^{\mathrm{op}}) \to \mathrm{Par}(\mathrm{G}),\ g \mapsto g\mathrm{P}g^{-1}$$

is an isomorphism onto an open subscheme of $\mathrm{Par}(\mathrm{G})$ which we denote by $\Omega(\mathrm{S},\mathrm{P})$. Let $\Phi(\mathrm{G},\mathrm{S})$ be the set of roots of G with respect to S. Since G is split, the choice of a special point o in $\mathrm{A}(\mathrm{S},k)$ determines a k°-Chevalley group $\mathscr{G}$ with generic fibre G. Any Chevalley basis in $\mathrm{Lie}(\mathscr{G})(k^\circ)$ leads to an isomorphism of $\mathrm{rad}^{\mathrm{u}}(\mathrm{P}^{\mathrm{op}})$ with the affine space

$$\prod_{\alpha\in\Psi} \mathrm{U}_\alpha \simeq \prod_{\alpha\in\Psi} \mathbb{A}^1_k,$$

where $\Psi = \Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{P}^{\mathrm{op}}),\mathrm{S}) = -\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{P}),\mathrm{S})$.

Proposition 1.14. — *We assume that the group* G *is split and we use the notation introduced above.*

(i) *The map ϑ_t sends the point o to the point of* $\Omega(\mathrm{S},\mathrm{P})^{\mathrm{an}}$ *corresponding to the multiplicative (semi)norm*

$$k[(\mathrm{X}_\alpha)_{\alpha\in\Psi}] \to \mathbb{R}_{\geqslant 0},\ \sum_{\nu\in\mathbb{N}^\Psi} a_\nu \mathrm{X}^\nu \mapsto \max_\nu |a_\nu|.$$

(ii) *Using the point o to identify the apartment* $\mathrm{A}(\mathrm{S},k)$ *with the vector space* $\Lambda(\mathrm{S}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(\mathrm{S}),\mathbb{R}_{>0})$*, the map* $\Lambda(\mathrm{S}) \to \mathrm{Par}(\mathrm{G})^{\mathrm{an}}$ *induced by ϑ_t associates with an element u of* $\Lambda(\mathrm{S})$ *the point of* $\Omega(\mathrm{S},\mathrm{P})^{\mathrm{an}}$ *corresponding to the multiplicative seminorm*

$$k[(\mathrm{X}_\alpha)_{\alpha\in\Psi}] \to \mathbb{R}_{\geqslant 0},\ \sum_{\nu\in\mathbb{N}^\Psi} a_\nu \mathrm{X}^\nu \mapsto \max_\nu |a_\nu| \prod_{\alpha\in\Psi} \langle u,\alpha\rangle^{\nu(\alpha)}.$$

(See [**RTW09**, Proposition 2.18].)

2. COMPARISON WITH GLUINGS

We show in this section that the compactifications defined in [**Wer07**] occur among the Berkovich compactifications. Let k be a non-Archimedean local field and let G be a connected semisimple k-group. We consider a faithful and geometrically irreducible linear representation $\rho : G \to GL_V$ of G. In [**Wer07**], a compactification $\overline{\mathscr{B}}(G,k)_\rho$ of the Bruhat-Tits building is constructed using the combinatorics of weights for ρ. It only depends on the Weyl chamber face position of the highest weight of the representation.

(2.1) We fix a maximal split torus S in G and denote by $\Phi = \Phi(G,S)$ the root system of G with respect to S. We denote by W the Weyl group of Φ and choose a W-invariant scalar product $(\cdot|\cdot)$ on the character group $X^*(S)$ of S, which we use to embed $X^*(S)$ in the vector space $\Lambda(S) = \mathrm{Hom}_{\mathbf{Ab}}(X^*(S), \mathbb{R}_{>0})$ via the map

$$X^*(S) \to \Lambda(S), \quad \chi \mapsto e^{(\chi|\cdot)}.$$

Let Δ be a basis of Φ. For every subset Y of Δ, we denote as in [**Wer07**] by P_Y^Δ the standard parabolic subgroup associated with Y; in particular, $P_\varnothing^\Delta$ is the minimal parabolic subgroup of G containing S and corresponding to Δ. The weights with respect to the action of S on V are called the k-weights of ρ. If T is a maximal torus containing S and if k'/k is a finite extension splitting T, then we have a natural projection

$$X^*(T \otimes_k k') \to X^*(S \otimes_k k') = X^*(S)$$

and there exists a basis Δ' of $\Phi(G \otimes_k k', T \otimes_k k')$ lifting Δ. With the basis Δ' is associated a well-defined character of $T \otimes_k k'$, the *highest weight* $\lambda_0(\Delta')$, whose restriction to S does not depend on any choice made for T, k' and Δ'. This character of S, denoted $\lambda_0(\Delta)$, is called the highest k-weight of ρ with respect to Δ; it defines an element in $\Lambda(S)$ lying in the Weyl cone $\mathfrak{C}(P_\varnothing^\Delta)$. Setting

$$Z = \{\alpha \in \Delta \,;\, (\lambda_0(\Delta)|\alpha) = 0\},$$

the linear subspace $\{\alpha = 1 \,;\, \alpha \in Z\}$ cuts out the only face of $\mathfrak{C}(P_\varnothing^\Delta)$ whose interior contains $\lambda_0(\Delta)$.

The purpose of this paragraph is to prove the following theorem.

Theorem 2.1. — *Let τ denote the type of the parabolic subgroup P_Z^Δ. The compactified buildings $\overline{\mathscr{B}}(G,k)_\rho$ and $\overline{\mathscr{B}}_\tau(G,k)$ are canonically isomorphic, and τ is the only k-rational type satisfying this condition.*

Remark 2.2. — Up to conjugacy, it is clear that the parabolic subgroup P_Z^Δ does not depend on the choice of S and Δ. Therefore, the k-rational type $t(P_Z^\Delta)$ is canonically associated with the absolutely irreducible representation ρ. One the other hand, the theory of highest weights of irreducible linear representations of split reductive groups singles out naturally a well-defined type $t(\rho)$ of parabolic subgroups of G, maybe non-k-rational: the connected component $\mathrm{Par}_{t(\rho)}(G)$ of $\mathrm{Par}(G)$ is characterized by the condition that, for any finite extension k'/k splitting G, this component contains all the maximal parabolic subgroups of G stabilizing a highest line in $V \otimes_k k'$ (see paragraph 4.1). We conclude this article by establishing that $t(P_Z^\Delta)$ is the unique k-rational type defining the same compactification of $\mathscr{B}(G,k)$ as the type $t(\rho)$ (cf. [**RTW09**, Appendix C]); equivalently, the compactification $\overline{\mathscr{B}}(G,k)_\rho$ defined in [**Wer07**] is canonically isomorphic to the Berkovich compactification $\overline{\mathscr{B}}_{t(\rho)}(G,k)$ (see Proposition 5.4).

Before proving this theorem, we can derive at once a comparison with the group-theoretic compactification [**GR06**].

Corollary 2.3. — *Let $t_{\min}$ be the type of a minimal parabolic subgroup of* G. *We denote by $\mathscr{V}_{\mathscr{B}(G,k)}$ the set of vertices in the Bruhat-Tits building $\mathscr{B}(G,k)$. Then the closure of $\mathscr{V}_{\mathscr{B}(G,k)}$ in the maximal*

Berkovich compactification $\overline{\mathscr{B}}_{t_{\min}}(\mathrm{G},k)$ *is* $\mathrm{G}(k)$*-equivariantly homeomorphic to the group-theoretic compactification of* $\mathscr{V}_{\mathscr{B}(\mathrm{G},k)}$.

Proof of corollary. By [**GR06**, Theorem 20], the group-theoretic compactification of $\mathscr{V}_{\mathscr{B}(\mathrm{G},k)}$ is $\mathrm{G}(k)$-equivariantly homeomorphic to the closure of $\mathscr{V}_{\mathscr{B}(\mathrm{G},k)}$ in the polyhedral compactification of $\mathscr{B}(\mathrm{G},k)$ defined by E. Landvogt. By [**Wer07**], we know that the latter compactification is $\mathrm{G}(k)$-equivariantly homeomorphic to $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$ where ρ is any weight lying in the interior of some Weyl chamber, i.e., such that $\mathrm{Z}=\varnothing$ with the notation above. Our claim follows from Theorem 2.1. □

Recall that every k-weight of ρ is of the form $\lambda_0(\Delta)-\sum_{\alpha\in\Delta}n_\alpha\alpha$ for certain non-negative integers n_α. We denote by $[\lambda_0(\Delta)-\lambda]=\{\alpha\in\Delta\,;\,n_\alpha>0\}$ the support of $\lambda_0(\Delta)-\lambda$. In [**Wer07**, Definition 1.1], a subset $\mathrm{Y}\subset\Delta$ is called *admissible*, if the set $\mathrm{Y}\sqcup\{\lambda_0(\Delta)\}$ is connected in the following sense: the graph with vertex set $\mathrm{Y}\cup\lambda_0(\Delta)$ and edges between all α and β such that $(\alpha|\beta)\neq 0$ is connected.

The following lemma is well-known, at least in characteristic 0 [**BT65**, 12.16]. It is a link between the abstract root-theoretic definition of admissibility, and its interpretation in terms of representations.

Lemma 2.4. — *A set* $\mathrm{Y}\subset\Delta$ *is admissible if and only if there exists a* k*-weight* μ *whose support* $[\lambda_0(\Delta)-\mu]$ *is equal to* Y.

Proof. For the sake of completeness, we show that this statement holds whatever the characteristic of k is. In order to be short, we freely use the notation of [**Bor91**, §24.B], which sums up the basic results of representation theory of reductive groups over arbitrary fields. In particular, given G as above, we denote by E^λ the unique Weyl G-module of highest weight λ and by F^λ its unique irreducible submodule (which in turn determines E^λ); in characteristic 0, we have $\mathrm{F}^\lambda=\mathrm{E}^\lambda$. Note that in the setting of this section, the G-module V is isomorphic to some F^λ and remains irreducible after extension of the ground field to the algebraic closure of k.

Let us first assume that Y is the support of some weight. Since the irreducible module F^λ is a submodule of the Weyl G-module E^λ, we deduce that Y is the support of some weight for E^λ. Moreover the Weyl module E^λ has the same character formula as the irreducible module of highest weight λ in characteristic 0, so the connectedness of the graph under consideration comes from the result in this case [**BT65**, 12.16]. Note that we use the classification of semisimple groups in order to find a group over a field of characteristic 0 having the same representations as G.

Conversely, let us assume that the graph $\mathrm{Y}\sqcup\{\lambda_0(\Delta)\}$ is connected. Recall that the set of weights is stable under the spherical Weyl group. We investigate first the case when Y is connected. We write $\mathrm{Y}=\{\beta_1,\beta_2,\dots,\beta_m\}$ in such a way that β_1 is connected to $\lambda_0(\Delta)$ (i.e., $(\lambda_0(\Delta)\mid\beta_1)\neq 0$) and that for any $i\leqslant m$ there exists $j<i$ such that β_i is connected to β_j (i.e., $(\beta_i\mid\beta_j)\neq 0$). Then it is easy to show by a finite induction on $l\leqslant m$, that the support of the weight $r_{\beta_l}r_{\beta_{l-1}}\dots r_{\beta_1}(\lambda_0(\Delta))$ is equal to $\{\beta_1,\beta_2,\dots,\beta_l\}$. Indeed, for $l=1$ this is clear since $r_{\beta_1}(\lambda_0(\Delta))=\lambda_0(\Delta)-2\frac{(\lambda_0(\Delta)|\beta_1)}{(\beta_1|\beta_1)}\beta_1$; and to pass from one step to the next one, we argue as follows. First, we have:

$$r_{\beta_l}r_{\beta_{l-1}}\dots r_{\beta_1}(\lambda_0(\Delta))=r_{\beta_l}\left(\lambda_0(\Delta)-\sum_{i=1}^{l-1}c_i\beta_i\right),$$

with each $c_i>0$ by induction hypothesis. This gives:

$$r_{\beta_l}r_{\beta_{l-1}}\dots r_{\beta_1}(\lambda_0(\Delta))=\lambda_0(\Delta)-\sum_{i=1}^{l-1}c_i\beta_i-2\left(\frac{(\lambda_0(\Delta)\mid\beta_l)}{(\beta_l|\beta_l)}-\sum_{i=1}^{l-1}c_i\frac{(\beta_i\mid\beta_l)}{(\beta_l|\beta_l)}\right)\beta_l,$$

which implies our claim by the numbering of the β_i's and the fact that $\lambda_0(\Delta)$ is dominant.

In the general case, we use a numbering $\mathrm{Y}_1,\mathrm{Y}_2,\dots\mathrm{Y}_s$ of the connected components of Y. The previous argument shows that there is a weight, say μ, with support equal to Y_1. Then we note that for each $\alpha\in\mathrm{Y}_1$ and each $\beta\in\mathrm{Y}_2$ we have $r_\beta(\alpha)=\alpha$. This allows us to apply the previous argument,

replacing $\lambda_0(\Delta)$ by μ and Y by Y_2. Our claim follows by induction on the number of connected components of Y. □

(2.2) For every admissible subset $Y \subset \Delta$ we set

$$Y^* = \{\alpha \in \Delta\,;\ (\alpha|\lambda_0(\Delta)) = 0 \text{ and } (\alpha|Y) = 0\}$$

and let C_Y^Δ denote the cone in $\Lambda(S)$ defined by the following conditions

$$\begin{cases} \alpha = 1, & \text{for all } \alpha \in Y \\ \lambda_0(\Delta) - \lambda \geqslant 1, & \text{for all } k\text{-weights } \lambda \text{ such that } [\lambda_0(\Delta) - \lambda] \not\subset Y. \end{cases}$$

Identifying the additive and multiplicative duals of $X^*(S)$ via the map $\mathbb{R} \to \mathbb{R}_{>0},\ x \mapsto e^x$, the cone $C_Y^\Delta \subset \Lambda(S)$ is the closure of the subset F_Y^Δ of $V(S) = \mathrm{Hom}_{\mathbf{Ab}}(X^*(S), \mathbb{R})$ defined in [**Wer07**, section 2]. It is shown in [loc. cit.] that $V(S)$ is the disjoint union of the subsets F_Y^Δ, where Y runs over the set of admissible subsets of Δ.

Lemma 2.5. — *Recall that $Z = \varnothing^*$ and let τ denote the type of the parabolic subgroup P_Z^Δ.*

(i) *A subset Y of Δ is admissible if and only if each of its connected components meets $\Delta - Z$.*

(ii) *For any admissible subset Y of Δ, we have*

$$C_Y^\Delta = C_\tau(P_Y^\Delta).$$

(iii) *The correspondence $Y \mapsto P_{Y \cup Y^*}^\Delta$ is a bijection between admissible subsets of Δ and τ-relevant parabolic subgroups containing $P_\varnothing^\Delta$.*

Proof. (i) This assertion is clear, since $Y \cup \{\lambda_0(\Delta)\}$ is connected if and only if each connected component of Y contains a root $\alpha \in \Delta$ with $(\alpha|\lambda_0(\Delta)) \neq 0$, i.e., a root in $\Delta - Z$.

(ii) Let Y be an admissible subset of Δ. It follows from (i) and from Proposition 1.8 (iii) that the linear space $\{\alpha = 1;\ \alpha \in Y\}$ cuts out a face of the cone $C_\tau(P_\varnothing^\Delta)$, namely the cone $C_\tau(P_Y^\Delta)$. Since this subspace cuts out the face C_Y^Δ of $C_\varnothing^\Delta$, it suffices to check that the cones $C_\tau(P_\varnothing^\Delta)$ and $C_\varnothing^\Delta$ coincide.

Let Δ' be another basis of the root system Φ. If $\lambda_0(\Delta') = \lambda_0(\Delta)$, then every x in the Weyl cone $\mathfrak{C}(P_\varnothing^{\Delta'})$ satisfies $(\lambda_0(\Delta) - \lambda)(x) \geqslant 1$ for all k-weights λ, hence $\mathfrak{C}(P_\varnothing^{\Delta'})$ is contained in $C_\varnothing^\Delta$. On the other hand, every point in the interior of $C_\varnothing^\Delta$ is contained in the Weyl cone $\mathfrak{C}(P_\varnothing^{\Delta'})$ for some basis Δ'. By [**Wer07**, Proposition 4.4 and Lemma 2.1], this implies $\lambda_0(\Delta') = \lambda_0(\Delta)$. Hence $C_\varnothing^\Delta$ is equal to the union of all Weyl cones $\mathfrak{C}(P_\varnothing^{\Delta'})$ with $\lambda_0(\Delta) = \lambda_0(\Delta')$. By definition, the cone $C_\tau(P_\varnothing^\Delta)$ is the union of all $\mathfrak{C}(P_\varnothing^{\Delta'})$ such that the minimal parabolic subgroup $P_\varnothing^{\Delta'}$ is contained in P_Z^Δ. Therefore, it remains to check that $\lambda_0(\Delta) = \lambda_0(\Delta')$, if and only if $P_\varnothing^{\Delta'}$ is contained in P_Z^Δ.

Let n be an element of $\mathrm{Norm}_G(S)(k)$ satisfying $nP_\varnothing^\Delta n^{-1} = P_\varnothing^{\Delta'}$, and let w be its image in the Weyl group W of Φ. Then $w(\Delta) = \Delta'$, hence $w(\lambda_0(\Delta)) = \lambda_0(\Delta')$. Besides, we have $nP_Z^\Delta n^{-1} = P_{w(Z)}^{\Delta'}$.

Assume that $\lambda_0(\Delta) = \lambda_0(\Delta')$. Then w fixes $\lambda_0(\Delta)$, which implies that $w(Z) = Z$ since the scalar product on $X^*(S)$ is W-invariant. Besides, for every $\alpha \in \Delta - Z$ there exists a k-weight λ such that $[\lambda_0(\Delta) - \lambda] = \{\alpha\}$ for $\{\alpha\}$ is an admissible subset of Δ. Since $w(\lambda)$ is a weight and $w(\lambda_0(\Delta)) = \lambda_0(\Delta)$, we deduce that $w(\alpha)$ is a positive root for Δ. Hence P_Z^Δ contains $P_\varnothing^{w(\Delta)} = P_\varnothing^{\Delta'}$.

Now assume that $P_\varnothing^{\Delta'} = nP_\varnothing^\Delta n^{-1}$ is contained in P_Z^Δ. Then n is contained in P_Z^Δ, which implies that w is in the Weyl group of the parabolic P_Z^Δ. Hence w is a product of reflections corresponding to roots in Z. Since roots in Z are perpendicular to $\lambda_0(\Delta)$, the corresponding reflections leave $\lambda_0(\Delta)$ invariant and therefore $\lambda_0(\Delta') = w(\lambda_0(\Delta)) = \lambda_0(\Delta)$.

(iii) Let Y be an admissible subset of Δ. By Proposition 1.8 (ii), the smallest τ-relevant parabolic subgroup containing P_Y^Δ is $P_{Y'}^\Delta$, where Y' is obtained by adjoining to Y all roots in Z which are perpendicular to each connected component of Y meeting $\Delta - Z$, hence to Y by (i). It follows that $Y' = Y \cup Y^*$. Conversely, if P_Z^Δ is a τ-relevant parabolic subgroup, then $C_\tau(P_Z^\Delta) = C_Y^\Delta$ for some admissible subset Y and $C_Y^\Delta = C(P_Y^\Delta)$ by (i). It follows from what we have just said that $P_{Y \cup Y^*}^\Delta$ is the

smallest τ-relevant parabolic subgroup containing P_Y^Δ, hence $C_\tau(P_Z^\Delta) = C_Y^\Delta = C_\tau(P_{Y\cup Y^*}^\Delta)$ and therefore $Z = Y \cup Y^*$. □

Thus, the fan consisting of all polyhedral cones C_Y^Δ coincides with the fan $\mathscr{F}_\tau$ defined in (1.4). Note that the type τ is non-degenerate since the representation ρ is faithful. Relying on [**RTW09**, Proposition B.3], it is not hard to check that the identity map of the apartment $A(S,k)$ extends to a homeomorphism j between the compactification $\overline{A}_\tau(S,k) = \overline{A(S,k)}^{\mathscr{F}_\tau}$ introduced in Definition 1.1 and the compactification $\overline{A(S,k)}_\rho$ of $A(S,k)$ defined from a different viewpoint in [**Wer07**, Sect. 2] (where it is simply denoted $\overline{A}$). This homeomorphism is compatible with the action of the group $\mathrm{Norm}_G(S)(k)$ on each space since this action is in both cases the unique continuous extension of the standard action of $\mathrm{Norm}_G(S)(k)$ on $A(S,k)$.

(2.3) Seen as a function $\Lambda(S) \to \mathbb{R}_{>0}$, each root $\alpha \in \Phi$ has a continuous extension $\widetilde{\alpha} : \overline{C} \to [0,\infty]$ for every cone C in the fan $\mathscr{F}_\tau$ over which either $\alpha \leqslant 1$ or $\alpha \geqslant 1$; this is obvious if we write $C = \mathrm{Hom}_{\mathbf{Mon}}(M,]0,1])$ and $\overline{C} = \mathrm{Hom}_{\mathbf{Mon}}(M, [0,1])$, where M is the saturated and finitely generated semigroup in $X^*(S)$ defined by

$$M = \{\alpha \in X^*(S)\ ;\ \alpha_{|C} \leqslant 1\}.$$

If $\tau = t_{\min}$ is the type of a minimal parabolic subgroup, then $\mathscr{F}_\tau$ is the Weyl fan and every root α satisfies $\alpha_{|C} \leqslant 1$ or $\alpha_{|C} \geqslant 1$ for each cone $C \in \mathscr{F}_{t_{\min}}$, hence extends continuously to the corresponding compactified vector space $\overline{\Lambda}(S)^{\mathscr{F}_{t_{\min}}}$. Since we have either $\alpha < 1$, $\alpha > 1$ or $\alpha = 1$ on the interior F° of each face F of $C \in \mathscr{F}_{t_{\min}}$, the extension $\widetilde{\alpha}$ of α to $\overline{C}$ satisfies

$$\begin{cases} \widetilde{\alpha}_{|C_F} = 0 & \text{if } \alpha_{|F^\circ} < 1, \\ 0 < \widetilde{\alpha}_{|C_F} < \infty & \text{if } \alpha_{|F} = 1, \\ \widetilde{\alpha}_{|C_F} = \infty & \text{if } \alpha_{|F^\circ} > 1, \end{cases}$$

where C_F is the stratum of $\overline{C}$ corresponding to the face F, namely the subset of $\overline{C}$ defined by the conditions

$$\begin{cases} \varphi = 0, & \text{for all } \varphi \in M \text{ such that } \varphi_{|F} \neq 1, \\ \varphi > 0, & \text{for all } \varphi \in M \text{ such that } \varphi_{|F} = 1. \end{cases}$$

This situation is illustrated by Figure 1 below with $G = \mathrm{SL}(3)$.

In general, we can always extend each root α to a *upper semicontinuous* function $\widetilde{\alpha} : \overline{\Lambda(S)}^{\mathscr{F}_\tau} \to [0,\infty]$ by setting

$$\widetilde{\alpha}(x) = \sup\{c \in \mathbb{R}_{>0}\ ;\ x \in \overline{\{\alpha \geqslant c\}}\}.$$

This function coincides with the continuous extension of $\alpha_{|C}$ to $\overline{C}$ for any cone C over which $\alpha \leqslant 1$ or $\alpha \geqslant 1$. In general, given a cone C and a face F of C, the upper semicontinuous extension $\widetilde{\alpha}$ of α to $\overline{C}$ satisfies

$$\begin{cases} \widetilde{\alpha}_{|C_F} = 0 & \text{if } \alpha_{|F^\circ} < 1 \\ 0 < \widetilde{\alpha}_{|C_F} < \infty & \text{if } \alpha_{|F} = 1 \\ \widetilde{\alpha}_{|C_F} = \infty & \text{if } \alpha_{|F^\circ} > 1 \\ \widetilde{\alpha}_{|C_F} = \infty & \text{otherwise.} \end{cases}$$

This follows easily from the existence of an affine function $\beta : C \to]0,1]$ such that $\beta_{|F} = 1$.

This situation is illustrated by Figure 2 below, where $G = \mathrm{SL}(3)$ and τ is a type of maximal proper parabolic subgroups.

With each point x of $\overline{A(S,k)}_\rho$ is associated in [**Wer07**] a subgroup P_x of $G(k)$ defined as follows. Set $N = \mathrm{Norm}_G(S)$ and recall that Bruhat-Tits theory provides us with a decreasing filtration $\{U_\alpha(k)_r\}_{r\in[-\infty,\infty]}$ on each unipotent root group $U_\alpha(k)$, with $U_\alpha(k)_{-\log(\infty)} = U_\alpha(k)_{-\infty} = U_\alpha(k)$ and $U_\alpha(k)_{-\log(0)} = U_\alpha(k)_\infty = \{1\}$. Then P_x is the subgroup of $G(k)$ generated by $N(k)_x = \{n \in N(k);\ nx = x\}$ and $U_\alpha(k)_{-\log\widetilde{\alpha}(x)}$ for all $\alpha \in \Phi$.

Let Q be a τ-relevant parabolic subgroup of G containing S and denote by L the Levi subgroup of Q associated with S. We consider the following decomposition of Φ in mutually disjoint closed subsets:

$$\Phi = \big(-\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}),\mathrm{S})\big) \cup \Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}),\mathrm{S}) \cup \Phi(\mathrm{L}',\mathrm{S}') \cup \Phi(\mathrm{L}'',\mathrm{S}''),$$

where L' and L'' are the normal and connected reductive subgroups of L such that the natural morphisms $\mathrm{L}' \times \mathrm{L}'' \to \mathrm{L}$ and $\mathrm{L}' \to \mathrm{Q}/\mathrm{R}_\tau(\mathrm{Q})$ are central isogenies, and where S' and S'' are the connected components of $\mathrm{S} \cap \mathrm{L}'$ and $\mathrm{S} \cap \mathrm{L}''$ respectively (see the discussion before Theorem 1.12). Equivalently, the subset $\Phi(\mathrm{L}',\mathrm{S}')$ of $\Phi(\mathrm{L},\mathrm{S})$ is the union of root systems $\Phi(\mathrm{H},\mathrm{S})$, where H runs over the set of quasi-simple components of L on which the restriction of τ is non-trivial, and $\Phi(\mathrm{L}'',\mathrm{S}'') = \Phi(\mathrm{L},\mathrm{S}) - \Phi(\mathrm{L}',\mathrm{S}')$.

Lemma 2.6. — *Let x be a point in the stratum $\Sigma = \mathrm{A}(\mathrm{S},k)/\langle \mathrm{C}_\tau(\mathrm{Q})\rangle$ of $\overline{\mathrm{A}(\mathrm{S},k)}^{\mathscr{F}_\tau}$.*

(i) *For any root α in Φ, we have:*

$$\begin{cases} \widetilde{\alpha}(x) = 0 \text{ and } \widetilde{-\alpha}(x) = \infty & \text{if } \alpha \in \Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}^{\mathrm{op}}),\mathrm{S}); \\ \widetilde{\alpha}(x) = \infty \text{ and } \widetilde{-\alpha}(x) = 0 & \text{if } \alpha \in -\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}^{\mathrm{op}}),\mathrm{S}); \\ \widetilde{\alpha}(x) = \widetilde{-\alpha}(x) = \infty & \text{if } \alpha \in \Phi(\mathrm{L}'',\mathrm{S}''); \\ 0 < \widetilde{\alpha}(x) < \infty & \text{if } \alpha \in \Phi(\mathrm{L}',\mathrm{S}'). \end{cases}$$

(ii) $\mathrm{P}_x = \mathrm{Stab}^t_{\mathrm{G}}(x)(k)$.

Proof. (i) This assertion follows from the identities

$$\Phi(\mathrm{rad}^{\mathrm{u}}(\mathrm{Q}^{\mathrm{op}}),\mathrm{S}) = \{\alpha \in \Phi\,;\ \alpha < 1 \text{ on the interior of } \mathrm{C}_\tau(\mathrm{Q})\},$$

$$\Phi(\mathrm{L}',\mathrm{S}') = \{\alpha \in \Phi\,;\ \alpha = 1 \text{ on } \mathrm{C}_\tau(\mathrm{Q})\}$$

and

$$\Phi(\mathrm{L}'',\mathrm{S}'') = \{\alpha \in \Phi\,;\ \alpha \text{ takes values } < 1 \text{ and } > 1 \text{ on } \mathrm{C}_\tau(\mathrm{Q})\}$$

(see Remark 1.9).

(ii) This assertion follows immediately from (i) and from the explicit description of $\mathrm{Stab}^t_{\mathrm{G}}(x)(k)$ in Theorem 1.12 since both P_x and $\mathrm{Stab}^t_{\mathrm{G}}(x)(k)$ are the subgroups of $\mathrm{G}(k)$ generated by $\mathrm{N}(k)_x$ and all $\mathrm{U}_\alpha(k)_{-\log\widetilde{\alpha}(x)}$, $\alpha \in \Phi$. □

The compactification $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$ defined in [**Wer07**] is the topological quotient of $\mathrm{G}(k) \times \overline{\mathrm{A}(\mathrm{S},k)}_\rho$ by the following equivalence relation:

$$(g,x) \sim (h,y) \Longleftrightarrow \big(\exists n \in \mathrm{N}(k),\ y = nx \ \text{ and } g^{-1}hn \in \mathrm{P}_x\big).$$

It follows immediately from assertion (ii) in the previous lemma and from the first assertion of Theorem 1.13 that the canonical homeomorphism

$$\mathrm{G}(k) \times \overline{\mathrm{A}(\mathrm{S},k)}^{\mathscr{F}_\tau} \ \xrightarrow{\sim}\ \mathrm{G}(k) \times \overline{\mathrm{A}(\mathrm{S},k)}_\rho$$

induces a $\mathrm{G}(k)$-homeomorphism between the compactified buildings $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ and $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$.

Uniqueness of the k-rational type τ such that the compactifications $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$ and $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ are isomorphic is easily checked. For any k-rational type τ' satisfying this condition, the compactifications $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ and $\overline{\mathscr{B}}_{\tau'}(\mathrm{G},k)$ are $\mathrm{G}(k)$-equivariantly homeomorphic. This homeomorphism identifies 0-dimensional strata; taking stabilizers in $\mathrm{G}(k)$, we obtain two parabolic subgroups P and P' of types τ and τ' respectively, which satisfy $\mathrm{P}(k) = \mathrm{P}'(k)$, hence $\mathrm{P} = \mathrm{P}'$ by Zariski density of rational points in parabolics and, finally, $\tau' = \tau$.

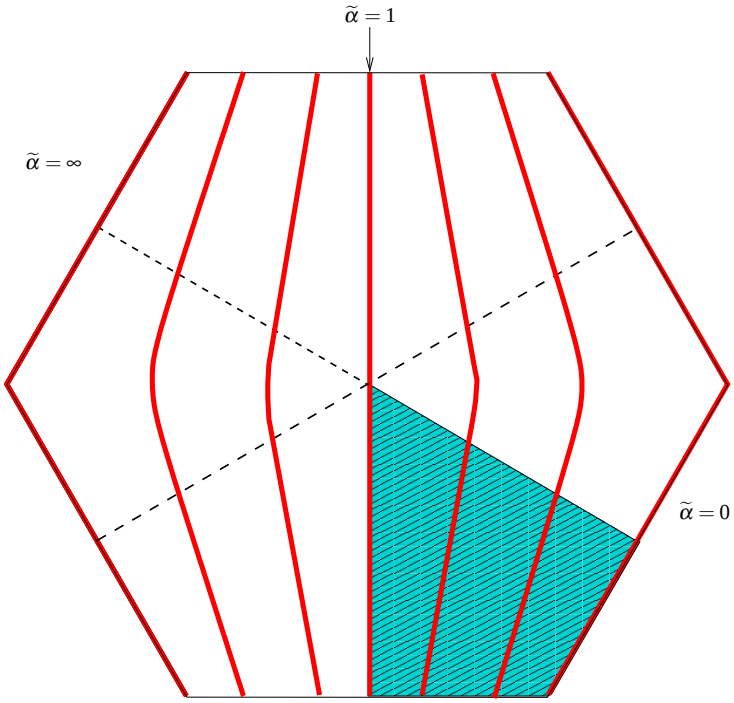

FIGURE 1. Compactified apartment in $\overline{\mathscr{B}}_{\varnothing}(\mathrm{SL}(3),k)$

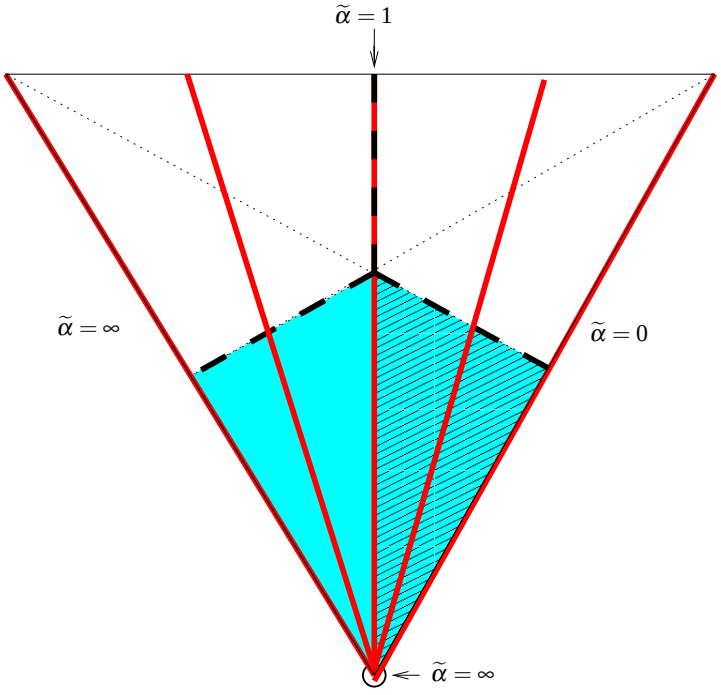

FIGURE 2. Compactified apartment in $\overline{\mathscr{B}}_{\tau}(\mathrm{SL}(3),k)$, with $\tau \neq \varnothing$

3. SEMINORM COMPACTIFICATION FOR GENERAL LINEAR GROUPS

We assume in this section that the non-Archimedean field k is discretely valued. In the following, we study a particular compactification of the building $\mathscr{B}(\mathrm{PGL_V},k)$ of $\mathrm{PGL_V}$, where V is a finite-dimensional k-vector space. From Berkovich's point of view, this is the compactification $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ associated with the flag variety $\mathrm{Par}_\delta(\mathrm{PGL_V})=\mathbb{P}(\mathrm{V})$, classifying flags of type $((0)\subset \mathrm{H}\subset \mathrm{V})$, where H is a hyperplane of V. One can give another description of this compactification as the projectivization of the cone of non-zero seminorms on V, thereby extending Goldman-Iwahori's construction of the building $\mathscr{B}(\mathrm{PGL_V},k)$. This compactification of $\mathscr{B}(\mathrm{PGL_V},k)$ should be seen as the non-Archimedean analogue of the projectivization of the cone of positive semidefinite hermitian matrices for a finite-dimensional complex vector space, the latter being the ambient space for Satake compactifications of symmetric spaces.

Starting with some reminder of Berkovich's note [**Ber95**] and of the third named author's paper [**Wer04**], we give an elementary description of the compactified building $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ and make everything explicit: convergence of seminorms, strata, stabilizers. An important feature of this compactification is the existence of a canonical retraction $\tau:\mathbb{P}(\mathrm{V})^{\mathrm{an}}\to\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$.

(3.1) Let $\mathrm{S}^\bullet\mathrm{V}$ be the symmetric algebra of the k-vector space V. This is a graded k-algebra of finite type whose spectrum (whose homogeneous spectrum, respectively) is the affine space $\mathbb{A}(\mathrm{V})$ (the projective space $\mathbb{P}(\mathrm{V})$, respectively):

$$\mathbb{A}(\mathrm{V})=\mathrm{Spec}(\mathrm{S}^\bullet\mathrm{V}) \text{ and } \mathbb{P}(\mathrm{V})=\mathrm{Proj}(\mathrm{S}^\bullet\mathrm{V}).$$

The underlying set of the k-analytic space $\mathbb{A}(\mathrm{V})^{\mathrm{an}}$ consists of all multiplicative seminorms on $\mathrm{S}^\bullet\mathrm{V}$. The underlying set of the k-analytic space $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ is the quotient of $\mathbb{A}(\mathrm{V})^{\mathrm{an}}-\{0\}$ by homothety: two non-zero seminorms x,y are equivalent if there exists a positive real number λ such that $|f|(y)=\lambda^n|f|(x)$ for any natural integer n and any element $f\in\mathrm{S}^n\mathrm{V}$.

Let $\mathscr{S}(\mathrm{V},k)$ be the set of all seminorms on the vector space V and let $\mathscr{X}(\mathrm{V},k)$ be the quotient of $\mathscr{S}(\mathrm{V},k)-\{0\}$ by homothety: two non-zero seminorms x and y on V are equivalent if there exists a positive real number $\lambda\in\mathbb{R}_{>0}$ such that $|f|(y)=\lambda|f|(x)$ for any $f\in\mathrm{V}$. Since each (multiplicative) seminorm on $\mathrm{S}^\bullet\mathrm{V}$ induces a seminorm on $\mathrm{V}=\mathrm{S}^1\mathrm{V}$ by restriction, we have a natural map $\tau:\mathbb{A}(\mathrm{V})^{\mathrm{an}}\to\mathscr{S}(\mathrm{V},k)$ such that $\tau(x)=0$ if and only if $x=0$. This map is obviously compatible with the above equivalence relations and therefore descends to a map $\tau:\mathbb{P}(\mathrm{V})^{\mathrm{an}}\to\mathscr{X}(\mathrm{V},k)$.

A seminorm x on the k-vector space V is *diagonalizable* if there exists a basis $(e_0,\dots,e_d)$ of V such that for every $v=\sum_{0\leqslant i\leqslant d}a_ie_i$ in V,

$$|v|(x)=\max_i|a_i||e_i|(x).$$

Proposition 3.1. — *Any non-zero seminorm on the k-vector space* V *is diagonalizable.*

Proof. As the absolute value of k is assumed to be discrete, this fact is established by F. Bruhat and J. Tits in [**BT84b**, Proposition 1.5 (i)]. It was initially proved by A. Weil in the locally compact case. □

Diagonalizability of seminorms on V allows us to define a canonical section j for both maps τ. Given a point x in $\mathscr{S}(\mathrm{V},k)-\{0\}$, pick a diagonalizing basis $(e_0,\dots,e_d)$ of V and consider the multiplicative seminorm defined on $\mathrm{S}^\bullet\mathrm{V}$ by

$$\sum_{v\in\mathbb{N}^d}\lambda_ve^v\mapsto\max_v|\lambda_v|\prod_{i=0}^d|e_i|(x)^{v_i}.$$

For any multiplicative seminorm z on $\mathrm{S}^\bullet\mathrm{V}$ inducing x on V, we have:

$$|e^v|(z)=\prod_{0\leqslant i\leqslant d}|e_i|(z)^{v_i}=\prod_{0\leqslant i\leqslant d}|e_i|(x)^{v_i},$$

hence

$$\left|\sum_\nu \lambda_\nu e^\nu\right|(z) \leqslant \max_\nu |\lambda_\nu||e^\nu|(z) = \max_\nu |\lambda_\nu| \prod_{i=0}^{d} |e_i|(x)^{\nu_i}.$$

Thus, the seminorm which we have just defined on $\mathrm{S}^\bullet\mathrm{V}$ is maximal among multiplicative seminorms on $\mathrm{S}^\bullet(\mathrm{V})$ inducing x on V and therefore it does not depend on the basis we picked; it will be denoted by $j(x)$. We also set $j(0) = 0$. The map $j : \mathscr{S}(\mathrm{V},k) \to \mathbb{A}(\mathrm{V})^{\mathrm{an}}$ so obtained is obviously a section of τ such that $j(x) = 0$ if and only if $x = 0$. Moreover, this map is compatible with above equivalence relations, hence descends to a map $j : \mathscr{X}(\mathrm{V},k) \to \mathbb{P}(\mathrm{V})^{\mathrm{an}}$ which is a section of τ.

Proposition 3.2. — (i) *For any points $x \in \mathscr{S}(\mathrm{V},k)$ and $z \in \mathbb{A}(\mathrm{V})^{\mathrm{an}}$ with $\tau(z) = x$, we have*

$$z \leqslant j(x).$$

(ii) *If we equip the sets $\mathscr{S}(\mathrm{V},k)$ and $\mathscr{X}(\mathrm{V},k)$ with the natural actions of the groups $\mathrm{GL_V}$ and $\mathrm{PGL_V}$ respectively, then the maps $j : \mathscr{S}(\mathrm{V},k) \to \mathbb{A}(\mathrm{V})^{\mathrm{an}}$ and $\tau : \mathbb{A}(\mathrm{V})^{\mathrm{an}} \to \mathscr{S}(\mathrm{V},k)$ ($j : \mathscr{X}(\mathrm{V},k) \to \mathbb{P}(\mathrm{V})^{\mathrm{an}}$ and $\tau : \mathbb{P}(\mathrm{V})^{\mathrm{an}} \to \mathscr{X}(\mathrm{V},k)$, respectively) are equivariant.*

Proof (i) We checked this inequality in the discussion above while defining j.

(ii) It is enough to prove that the maps $j : \mathscr{S}(\mathrm{V},k) \to \mathbb{A}(\mathrm{V})^{\mathrm{an}}$ and $\tau : \mathbb{A}(\mathrm{V})^{\mathrm{an}} \to \mathscr{S}(\mathrm{V},k)$ are $\mathrm{GL_V}(k)$-equivariant. This is trivially true for τ since this map sends a seminorm on $\mathrm{S}^\bullet\mathrm{V}$ to its restriction to $\mathrm{V} = \mathrm{S}^1\mathrm{V}$. For any elements $x \in \mathscr{S}(\mathrm{V},k) - \{0\}$ and $g \in \mathrm{GL_V}(k)$, the point $z = g^{-1}j(gx)$ of $\mathbb{A}(\mathrm{V})^{\mathrm{an}}$ satisfies $\tau(z) = g^{-1}\tau j(gx) = g^{-1}gx = x$, hence $g^{-1}j(gx) \leqslant j(x)$ according to (i). Substituting gx to x and g to g^{-1} in this inequality, we obtain $gj(x) = gj(g^{-1}gx) \leqslant j(gx)$ and therefore $j(gx) = gj(x)$. □

In the special case of the semisimple group $\mathrm{PGL_V}$ and of the flag variety $\mathbb{P}(\mathrm{V}) = \mathrm{Par}_\delta(\mathrm{PGL_V})$, where δ is the type of parabolic subgroups stabilizing a hyperplane in V, this elementary picture provides us with an alternative description of the general construction of [**RTW09**, 2.4], recalled in section 1. We thus recover the classical realization of the building $\mathscr{B}(\mathrm{PGL_V},k)$ as the space of norms on V up to homothety ([**GI63**], [**BT84b**]) and the construction of a compactification in terms of seminorms [**Wer04**].

Proposition 3.3. — *There exists one and only one map $\iota : \overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k) \to \mathscr{X}(\mathrm{V},k)$ such that the diagram*

$$\begin{array}{ccc} \overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k) & \xrightarrow{\vartheta_\delta} & \mathrm{Par}_\delta(\mathrm{PGL_V},k)^{\mathrm{an}} \\ \downarrow{\scriptstyle \iota} & & \| \\ \mathscr{X}(\mathrm{V},k) & \xrightarrow[j]{} & \mathbb{P}(\mathrm{V})^{\mathrm{an}} \end{array}$$

is commutative. This map has the following properties:

(i) *it is bijective and $\mathrm{PGL_V}$-equivariant;*

(ii) *it identifies $\mathscr{B}(\mathrm{PGL_V},k)$ with the subset of $\mathscr{X}(\mathrm{V},k)$ consisting of all homothety classes of norms on* V*; more generally, given a subspace* W *of* V*, ι identifies the stratum $\mathscr{B}(\mathrm{V/W},k)$ of $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ with the subset of $\mathscr{X}(\mathrm{V},k)$ consisting of all homothety classes of seminorms on* V *with kernel* W*;*

(iii) *for any maximal split torus* T *in* $\mathrm{PGL_V}$*, the map ι identifies the compactified apartment $\overline{\mathrm{A}}_\delta(\mathrm{T},k)$ in $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ with the set of homothety classes of* T*-diagonalizable seminorms on* V *(i.e., seminorms which are diagonalizable in a basis of* V *consisting of eigenvectors for the maximal split torus in* $\mathrm{GL_V}$ *lifting* T*).*

Proof. If it exists, such a map ι is unique since j is injective.

The existence of ι follows easily from the explicit description of the map ϑ_δ recalled in (1.7). Pick a maximal split torus T in $\mathrm{PGL_V}$ and a basis $(e_0,\dots,e_d)$ of V consisting of eigenvectors for the

maximal split torus in $\mathrm{GL_V}$ lifting T. Using Proposition 1.14, one sees that the map ϑ_δ realizes a bijection between the compactified apartment $\overline{\mathrm{A}}_\delta(\mathrm{T},k)$ and the subset of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ consisting of homothety classes of all multiplicative seminorms x on $\mathrm{S}^\bullet\mathrm{V}$ satisfying the following condition: there exist non-negative real numbers $c_0,\dots,c_d$, not all equal to zero, such that $|\sum_\nu \lambda_\nu e^\nu|(x) = \max_\nu |\lambda_\nu| \prod_{0\leqslant i\leqslant d} c_i^{\nu_i}$. The subset $\vartheta_\delta\left(\overline{\mathrm{A}}_\delta(\mathrm{T},k)\right)$ of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ is therefore the image under j of the subset $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$ of $\mathscr{X}(\mathrm{V},k)$ consisting of homothety classes of all T-diagonalizable seminorms on V (i.e., diagonalizable by the split maximal torus of $\mathrm{GL_V}$ lifting T). Since $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ is the union of all compactified apartments associated with maximal tori in $\mathrm{PGL_V}$, the image of the map ϑ_δ is therefore contained in the image of j. This observation establishes the existence of the application ι; it also proves (iii).

The map ι is injective, because so is ϑ_δ. Surjectivity follows from the fact that $\mathscr{X}(\mathrm{V},k)$ is the union of the subsets $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$, where T runs over the set of maximal split tori in $\mathrm{PGL_V}$. To see that the map ι is $\mathrm{PGL_V}(k)$-equivariant, it suffices to observe that ι is the composition $\tau\vartheta_\delta$ of two equivariant maps. Indeed, since $\tau j = \mathrm{id}_{\mathscr{X}(\mathrm{V},k)}$,

$$j\tau\vartheta_\delta = j\tau j\iota = j\iota$$

and thus $\tau\vartheta_\delta = \iota$.

We now check (ii). Let W be a linear subspace of V and consider a seminorm x on V. The point $j(x)$ in $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ belongs to the subspace $\mathbb{P}(\mathrm{V/W})^{\mathrm{an}}$ of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ if and only if the seminorm $j(x) : \mathrm{S}^\bullet\mathrm{V} \to \mathbb{R}_{\geqslant 0}$ factors through the canonical homomorphism $\mathrm{S}^\bullet\mathrm{V} \to \mathrm{S}^\bullet(\mathrm{V/W})$. By multiplicativity, this is the case if and only if x vanishes identically on W. Since the stratum $\mathscr{B}(\mathrm{PGL_{V/W}},k)$ of $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ is the preimage under ϑ_δ of the space

$$\mathbb{P}(\mathrm{V/W})^{\mathrm{an}} - \bigcup_{\mathrm{W}\subsetneq\mathrm{W}'\subsetneq\mathrm{V}} \mathbb{P}(\mathrm{V/W}')^{\mathrm{an}},$$

we conclude that ι identifies this stratum with the subspace of $\mathscr{X}(\mathrm{V},k)$ consisting of homothety classes of seminorms on V with kernel W; in particular, this map is a bijection between $\mathscr{B}(\mathrm{PGL_V},k)$ and the set of homothety classes of *norms* on V. □

We can introduce a natural topology on $\mathscr{X}(\mathrm{V},k)$: equip the set $\mathscr{S}(\mathrm{V},k)$ with the coarsest topology such that each evaluation map $(x \mapsto |v|(x),\ v\in\mathrm{V})$ is continuous and consider the quotient topology on $\mathscr{X}(\mathrm{V},k)$. The map $\tau : \mathbb{P}(\mathrm{V})^{\mathrm{an}} \to \mathscr{X}(\mathrm{V},k)$ is obviously continuous. If the field k is locally compact, then the map $j : \mathscr{X}(\mathrm{V},k) \to \mathbb{P}(\mathrm{V})^{\mathrm{an}}$ is continuous (see point (ii) below).

Proposition 3.4. — *The set $\mathscr{X}(\mathrm{V},k)$ is equipped with the topology which we have just defined.*

(i) *The map $\iota : \overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k) \to \mathscr{X}(\mathrm{V},k)$ is continuous and, for any maximal split torus* T *in* $\mathrm{PGL_V}$, *it induces a homeomorphism between the compactified apartment $\overline{\mathrm{A}}_\delta(\mathrm{T},k)$ and the subspace $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$ of $\mathscr{X}(\mathrm{V},k)$ consisting of homothety classes of* T*-diagonalizable seminorms on* V.

(ii) *If k is locally compact, the map ι is a homeomorphism and the map $j : \mathscr{X}(\mathrm{V},k) \to \mathbb{P}(\mathrm{V})^{\mathrm{an}}$ is a homeomorphism onto its image.*

Proof. (i) Continuity of ι is obvious if we write this map as the composition $\tau\vartheta_\delta$. Given a maximal split torus T in $\mathrm{PGL_V}$, the map ι induces a continuous bijection between the compact space $\overline{\mathrm{A}}_\delta(\mathrm{T})$ and its image in $\mathscr{X}(\mathrm{V},k)$; this map is a homeomorphism since the topological space $\mathscr{X}(\mathrm{V},k)$ is Hausdorff.

(ii) If the field k is locally compact, the topological space $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ is compact and the continuous bijection ι onto the Hausdorff topological space $\mathscr{X}(\mathrm{V},k)$ is a homeomorphism. The map ϑ_δ is a homeomorphism onto its image; writing the map j as the composition $\vartheta_\delta\iota^{-1}$, we see that the same is true for j. □

The topology which we consider on $\mathscr{X}(\mathrm{V},k)$ is relevant only if the field k is locally compact. In general, we have to modify it and endow $\mathscr{X}(\mathrm{V},k)$ with the topology deduced from $\overline{\mathscr{B}}_\delta(\mathrm{PGL_V},k)$ via the bijection ι. Equivalently, pick a maximal split torus T in $\mathrm{PGL_V}$, endow $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$ with the coarsest topology such that all evaluations $(x \mapsto |v|(x),\ v\in\mathrm{V})$ are continuous and equip $\mathscr{X}(\mathrm{V},k)$

with the quotient topology deduced from the surjective map

$$\mathrm{G}(k)\times\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)\to\mathscr{X}(\mathrm{V},k),\ (g,x)\mapsto g\cdot x.$$

The above identification between $\overline{\mathscr{B}}_{\delta}(\mathrm{PGL}_{\mathrm{V}},k)$ and $\mathscr{X}(\mathrm{V},k)$ allows us to describe the subgroup of $\mathrm{PGL}_{\mathrm{V}}$ fixing a given point x of $\mathscr{X}(\mathrm{V},k)$. Let W be the kernel of x and let P be the parabolic subgroup of $\mathrm{PGL}_{\mathrm{V}}$ stabilizing W. The subgroup of $\mathrm{PGL}_{\mathrm{V}}(k)(k)$ fixing x is contained in $\mathrm{P}(k)$; this is the extension of the maximal bounded subgroup of $\mathrm{PGL}_{\mathrm{V/W}}(k)$ fixing the norm (induced by) x on V/W by the subgroup of $\mathrm{P}(k)$ acting trivially on W.

More explicitly, if $(e_0,\dots,e_d)$ is a basis of V diagonalizing x and chosen so that $\mathrm{W}=\mathrm{Span}(e_m,\dots,e_d)$, then $\mathrm{P}(k)$ is the subgroup of lower triangular block matrices

$$\begin{pmatrix}\mathrm{GL}(m,k) & 0\\ * & \mathrm{GL}(d+1-m,k)\end{pmatrix}$$

modulo homothety. Moreover, if the basis can be chosen so that x satisfies $|e_i|(x)=1$ for any $i\in\{0,\dots,m-1\}$, i.e., if x is a vertex of $\mathscr{X}(\mathrm{V/W},k)$, then its stabilizer in $\mathrm{PGL}_{\mathrm{V}}(k)$ is a conjugate of the subgroup of matrices

$$\begin{pmatrix}k^{\times}\cdot\mathrm{GL}(m,k^{\circ}) & 0\\ * & \mathrm{GL}(d+1-m,k)\end{pmatrix}$$

modulo homothety.

(3.2) Assuming that the field k is locally compact, we complete our description of $\mathscr{X}(\mathrm{V},k)\cong\overline{\mathscr{B}}_{\delta}(\mathrm{PGL}_{\mathrm{V}},k)$ in terms of seminorms. We fix a basis $(e_0,\dots,e_d)$ of V and denote by T and $\widetilde{\mathrm{T}}$ the corresponding split maximal tori in $\mathrm{PGL}_{\mathrm{V}}$ and GL_{V} respectively. We also denote by o the norm on V defined by

$$\left|\sum_{i=0}^{d}a_ie_i\right|(o)=\max_{0\leqslant i\leqslant d}|a_i|$$

and set

$$\mathrm{K}(o)=\{g\in\mathrm{GL}_{\mathrm{V}}(k)\ ;\ g\cdot o=o\}.$$

Proposition 3.5. — *A complete set of representatives for the action of* $\mathrm{GL}_{\mathrm{V}}(k)$ *on* $\mathscr{S}(\mathrm{V},k)-\{0\}$ *consists of all non-zero* $\widetilde{\mathrm{T}}$*-diagonalizable seminorms* x *on* V *satisfying* $0\leqslant|e_d|(x)\leqslant\ldots\leqslant|e_1|(x)\leqslant|e_0|(x)\leqslant q$*, where* $q>1$ *generates the group* $|k^{\times}|$.

(ii) The set $\overline{\mathscr{C}}$ *of non-zero* $\widetilde{\mathrm{T}}$*-diagonalizable seminorms* x *on* V *satisfying* $0\leqslant|e_d|(x)\leqslant\ldots\leqslant|e_1|(x)\leqslant|e_0|(x)$ *is a fundamental domain for the* $\mathrm{K}(o)$*-action on* $\mathscr{S}(\mathrm{V},k)-\{0\}$.

Proof. (i) Since each seminorm on V is diagonalizable by some maximal split torus, it follows from conjugacy of maximal split tori that each orbit of $\mathrm{GL}_{\mathrm{V}}(k)$ in $\mathscr{S}(\mathrm{V},k)-\{0\}$ meets the set $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ of non-zero $\widetilde{\mathrm{T}}$-diagonalizable seminorms.

Let ϖ be a generator of the maximal ideal of k°, i.e., $|\varpi|=q^{-1}<1$ generates $|k^{\times}|$, and pick $\nu\in\mathbb{N}^{d+1}$. By definition of the GL_{V}-action on $\mathscr{S}(\mathrm{V},k)-\{0\}$, $\mathrm{diag}(\varpi^{\nu})\cdot o$ is the $\widetilde{\mathrm{T}}$-diagonalizable seminorm on V such that

$$|e_i|(\mathrm{diag}(\varpi^{\nu})\cdot o)=|\mathrm{diag}(\varpi^{-\nu})\cdot e_i|(o)=|\varpi^{-\nu_i}e_i|(o)=q^{\nu_i}.$$

Accordingly, for any permutation $w\in\mathfrak{S}_{d+1}$ the permutation matrix $n(w)$ maps a $\widetilde{\mathrm{T}}$-diagonalizable seminorm x to the $\widetilde{\mathrm{T}}$-diagonalizable seminorm $n(w)\cdot x$ satisfying

$$|e_i|(n(w)\cdot x)=|n(w)^{-1}\cdot e_i|(x)=|e_{w^{-1}(i)}|(x).$$

Combining these two observations, one checks immediately that each $\mathrm{GL}_{\mathrm{V}}(k)$-orbit in $\mathscr{S}(\mathrm{V},k)-\{0\}$ meets the subset of $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ consisting of seminorms x such that

$$0\leqslant|e_d|(x)\leqslant\ldots\leqslant|e_1|(x)\leqslant|e_0|(x)\leqslant q.$$

(ii) As in (i), one easily shows that any $\mathrm{K}(o)$-orbit meets $\overline{\mathscr{C}}$.

For any point x in $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$, we can extend x to a seminorm on the exterior algebra $\Lambda^{\bullet}\mathrm{V}$ as follows: defining as usual e_{I} as the product $e_{i_1}\wedge\ldots\wedge e_{i_m}$ for any subset $\mathrm{I}=\{i_1,\ldots,i_m\}$ of $\{0,\ldots,d\}$ with $i_1<\ldots<i_m$, we set

$$|e_{\mathrm{I}}|(x)=\prod_{i\in\mathrm{I}}|e_i|(x)\quad\text{and}\quad\left|\sum_{\mathrm{I}}a_{\mathrm{I}}e_{\mathrm{I}}\right|(x)=\max_{\mathrm{I}}|a_{\mathrm{I}}|\cdot|e_{\mathrm{I}}|(x).$$

Pick x in $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ and assume that we have $g\cdot x\in\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ for some $g\in\mathrm{K}(o)$. If we use the basis $(e_0,\ldots,e_d)$ to identify V with k^{d+1}, then $\mathrm{K}(o)$ is the subgroup $\mathrm{GL}(d+1,k^{\circ})$ of $\mathrm{GL}(d+1,k)$. For each $m\in\{1,\ldots,d\}$, this observation implies immediately

$$\max_{\mathrm{I}}|e_{\mathrm{I}}|(g\cdot x)=\max_{\mathrm{I}}|\Lambda^m g^{-1}\cdot e_{\mathrm{I}}|(x)=\max_{\mathrm{I}}|e_{\mathrm{I}}|(x),$$

where the maximum is taken over all subsets $\mathrm{I}\subset\{0,\ldots,d\}$ of cardinality m. If we assume that both x and $g\cdot x$ belong to $\overline{\mathscr{C}}$, it follows recursively that $|e_i|(g\cdot x)=|e_i|(x)$ for any $i\in\{0,\ldots,d\}$, hence $g\cdot x=x$. Therefore, each $\mathrm{K}(o)$-orbit contains a unique point lying in $\overline{\mathscr{C}}$. □

Convergence of seminorms up to homothety. We examine now the convergence of sequences in $\mathscr{X}(\mathrm{V},k)$, from which one can recover that $\mathscr{X}(\mathrm{V},k)$ is a compactification of the Bruhat-Tits building $\mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$.

Let (z_n) be a sequence of $\widetilde{\mathrm{T}}$-diagonalizable seminorms. We say that this sequence is *normalized from below* if $|e_i|(z_n)\geqslant 1$ for all $i\in\{0,\ldots,d\}$ and all $n\geqslant 0$ such that $|e_i|(z_n)\neq 0$. Furthermore, we say that (z_n) is *distinguished* if there exists a non-empty subset I of $\{0,\ldots,d\}$ such that:

(a) for any $i,j\in\mathrm{I}$, the sequence $\left(\frac{|e_i|(z_n)}{|e_j|(z_n)}\right)_n$ converges to a positive real number;

(b) for any $i\in\mathrm{I}$ and $j\in\{0,\ldots,d\}-\mathrm{I}$, the sequence $\left(\frac{|e_j|(z_n)}{|e_i|(z_n)}\right)_n$ converges to 0.

In this situation, we set $\frac{|e_i|(z_\infty)}{|e_j|(z_\infty)}=\lim_n\left(\frac{|e_i|(z_n)}{|e_j|(z_n)}\right)$ for any $i,j\in\mathrm{I}$ and we say that I is the *index set at infinity* of the sequence (z_n).

The following proposition describes the convergence of sequences in $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$. We recall that $\overline{\mathscr{C}}$ denotes the subset of $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ consisting of seminorms x satisfying $0\leqslant|e_d|(x)\leqslant\ldots\leqslant|e_1|(x)\leqslant|e_0|(x)$.

Proposition 3.6. — *Let (x_n) be a sequence of points in $\mathscr{X}_{\mathrm{T}}(\mathrm{V},k)$.*

(i) *Up to going over to a subsequence, there exists a sequence (z_n) in $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ lifting (x_n) and an element w of $\mathfrak{S}_{d+1}$ such that the sequence $(n(w)z_n)$ is normalized from below, distinguished and contained in $\overline{\mathscr{C}}$.*

(ii) *Assume that (x_n) comes from a sequence (z_n) of points in $\overline{\mathscr{C}}$ normalized from below and distinguished, with index set at infinity* I. *We have $\lim(x_n)=x_\infty$, where x_∞ is the homothety class of the $\widetilde{\mathrm{T}}$-diagonalizable seminorms z_∞^{I} defined by picking an element i of* I *and setting*

$$|e_j|(z_\infty^{\mathrm{I}})=\begin{cases}\frac{|e_j|(z_\infty)}{|e_i|(z_\infty)} & \text{if } j\in\mathrm{I};\\ 0 & \text{if } j\notin\mathrm{I}.\end{cases}$$

(iii) *The topological space $\mathscr{X}(\mathrm{V},k)$ is metrizable and compact. It contains the Bruhat-Tits building of $\mathrm{PGL}_{\mathrm{V}}(k)$ as a dense open subset.*

Proof. (i) Let (z_n) be any sequence in $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ lifting (x_n). The seminorm z_n is non-zero, so the real number μ_n, defined as the minimum of the finite set $\{|e_i|(z_n)\ ;\ 0\leqslant i\leqslant d\ \text{ and }|e_i|(z_n)\neq 0\}$, is positive. For $\lambda_n=\mu_n^{-1}$, the sequence $\{\lambda_n\cdot z_n\}_{n\geqslant 0}$ is normalized from below. It is therefore enough to show that any sequence in $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ which is normalized from below admits a distinguished subsequence, up to multiplication by a permutation matrix $n(w)$. For simplicity, let us denote again by (z_n) such a sequence.

For each $n \geqslant 0$, there exists $i_n \in \{0,1,\dots,d\}$ such that $|e_{i_n}|(z_n) = \max_{0\leqslant i\leqslant d}\{|e_i|(z_n)\}$. The sequence $(i_n)_n$ takes its values in a finite set, so up to extracting, we may assume that it is constant. By iterating the same argument, we find $w \in \mathfrak{S}_{d+1}$ such that:

$$|e_{w(0)}|(z_n) \geqslant |e_{w(1)}|(z_n) \geqslant \ldots \geqslant |e_{w(d)}|(z_n)$$

for any $n \geqslant 0$, that is such that the sequence $(n(w^{-1})\cdot z_n)$ lies in $\overline{\mathscr{C}}$.

Note that since (z_n) is normalized from below, we have $|e_{w(0)}|(z_n) \geqslant 1$ for each $n \geqslant 0$. For each $i \in \{0,1,\dots,d\}$, let us set $\beta_i = \limsup_n \frac{|e_{w(i)}|(z_n)}{|e_{w(0)}|(z_n)}$; we have: $1 = \beta_0 \geqslant \beta_1 \geqslant \ldots \geqslant \beta_d \geqslant 0$. Up to extracting, we may assume that $\lim_n \left(\frac{|e_{w(i)}|(z_n)}{|e_{w(0)}|(z_n)}\right) = \beta_i$ for each i. Define I as the subset of $\{0,\dots,d\}$ consisting of indices i such that $\beta_i > 0$; note that I contains 0 by assumption, hence is non-empty. For any $i,j \in \mathrm{I}$, the sequence

$$\frac{|e_{w(i)}|(z_n)}{|e_{w(j)}|(z_n)} = \frac{|e_{w(i)}|(z_n)}{|e_{w(0)}|(z_n)} \cdot \frac{|e_{w(0)}|(z_n)}{|e_{w(j)}|(z_n)}$$

converges to the positive real number $\frac{\beta_i}{\beta_j}$, whereas for any $i \in \mathrm{I}$ and $j \in \{0,\dots,d\} - \mathrm{I}$ the sequence

$$\frac{|e_{w(j)}|(z_n)}{|e_{w(i)}|(z_n)} = \frac{|e_{w(j)}|(z_n)}{|e_{w(0)}|(z_n)} \cdot \frac{|e_{w(0)}|(z_n)}{|e_{w(i)}|(z_n)}$$

converges to $\frac{\beta_j}{\beta_i} = 0$. Thus, the sequence $(n(w^{-1})\cdot z_n)$ is distinguished.

(ii) Let (z_n) be a sequence in $\mathscr{S}_{\widetilde{\mathrm{T}}}(\mathrm{V},k)$ lifting (x_n), which we assume to be normalized from below and distinguished. Let I denote its index set at infinity. Since

$$\frac{|e_\ell|(z^i_\infty)}{|e_\ell|(z^j_\infty)} = \frac{|e_\ell|(z_\infty)}{|e_i|(z_\infty)} \cdot \frac{|e_j|(z_\infty)}{|e_\ell|(z_\infty)} = \frac{|e_j|(z_\infty)}{|e_i|(z_\infty)}$$

for any $i,j,\ell \in \mathrm{I}$, the $\widetilde{\mathrm{T}}$-diagonalizable seminorms z^i_∞ and z^j_∞ define the same homothety class in $\mathscr{X}(\mathrm{V},k)$. Given $i \in \mathrm{I}$, the seminorm $y^i_n = |e_i|(z_n)^{-1}\cdot z_n$ satisfies

$$\lim_n |e_\ell|(y^i_n) = \lim_n \frac{|e_\ell|(z_n)}{|e_i|(z_n)} = \begin{cases} \frac{|e_\ell|(z_\infty)}{|e_i|(z_\infty)} & \text{if } \ell \in \mathrm{I} \\ 0 & \text{if } \ell \in \{0,\dots,d\} - \mathrm{I} \end{cases}$$

since the sequence (z_n) is distinguished and thus the sequence (y^i_n) converges to the seminorm z^i_∞ in $\mathscr{S}(\mathrm{V},k) - \{0\}$.

(iii) Let k_0 denote a dense and countable subfield of k and let V_0 be a k_0-vector subspace of V such that $\mathrm{V} = \mathrm{V}_0 \otimes_{k_0} k$; this is a dense and countable subset of V. Each non-zero seminorm on V is completely determined by its restriction to V_0, hence the map

$$\mathscr{S}(\mathrm{V},k) \to \mathbb{R}^{\mathrm{V}_0},\ x \mapsto (v \mapsto |v|(x))$$

is a continuous injection. Since $\mathscr{S}(\mathrm{V},k)$ is locally compact, this injection is a homeomorphism of $\mathscr{S}(\mathrm{V},k)$ onto its image. This map induces a homeomorphism of $\mathscr{X}(\mathrm{V},k)$ onto a subspace of $\mathbb{R}^{\mathrm{V}_0}/\mathbb{R}_{>0}$ and, since the latter topological space is metrizable, so is $\mathscr{X}(\mathrm{V},k)$.

It follows from (ii) that the image $\overline{\mathscr{Q}}$ of $\overline{\mathscr{C}}$ in $\mathscr{X}(\mathrm{V},k)$ is compact. The map $\pi : \mathrm{K}(o) \times \overline{\mathscr{Q}} \to \mathscr{X}(\mathrm{V},k)$ induced by the $\mathrm{GL}_{\mathrm{V}}(k)$-action is continuous, and it is surjective by Proposition 3.5, (ii). Since $\mathrm{K}(o) \simeq \mathrm{GL}_{d+1}(k^\circ)$, the source is compact; as the target is Hausdorff, compactness of $\mathscr{X}(\mathrm{V},k)$ follows.

Identifying the Bruhat-Tits building $\mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$ with the subspace of $\mathscr{X}(\mathrm{V},k)$ consisting of classes of norms on V, the complementary subspace $\mathscr{X}(\mathrm{V},k) - \mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k) = \mathrm{K}(o)\cdot\left[\overline{\mathscr{Q}} \cap (\mathscr{X}(\mathrm{V},k) - \mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k))\right]$ is closed and therefore $\mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$ is open in $\mathscr{X}(\mathrm{V},k)$. Density is obvious. □

Orbit structure. We have already observed in Proposition 3.3 that the canonical identification $\mathscr{X}(\mathrm{V},k) \cong \overline{\mathscr{B}}_\delta(\mathrm{PGL}_\mathrm{V},k)$ transforms the natural stratification of $\overline{\mathscr{B}}_\delta(\mathrm{PGL}_\mathrm{V},k)$ into the stratification of $\mathscr{X}(\mathrm{V},k)$ by kernels: with each point x of $\mathscr{X}(\mathrm{V},k)$ is associated the non-zero linear subspace $\mathrm{V}(x) = \{v \in \mathrm{V}\,;\ |v|(x) = 0\}$ and two points $x,y \in \mathscr{X}(\mathrm{V},k)$ belong to the same stratum if $\mathrm{V}(x) = \mathrm{V}(y)$. The set of strata is indexed by the set of non-zero linear subspaces of V and the stratum associated with a linear subspace W is canonically isomorphic to the building $\mathscr{B}(\mathrm{PGL}_{\mathrm{V}/\mathrm{W}},k)$.

Given any point x of $\mathscr{X}(\mathrm{V},k)$, its stabilizer in $\mathrm{PGL}_\mathrm{V}(k)$ is the extension of a maximal compact subgroup of $\mathrm{PGL}_{\mathrm{V}/\mathrm{V}(x)}(k)$ by $\mathrm{PGL}_{\mathrm{V}(x)}(k)$, and its Zariski closure is the parabolic subgroup fixing $\mathrm{V}(x)$.

All these assertions can be easily proved starting from the definition of $\mathscr{X}(\mathrm{V},k)$, without knowing the structure of the Berkovich compactification $\overline{\mathscr{B}}_\delta(\mathrm{PGL}_\mathrm{V},k)$. One can also show that the unique closed orbit for the PGL_V-action on $\mathscr{X}(\mathrm{V},k)$ consists of the homothety classes of seminorms of the form $|.| \circ \varphi$, where φ is a non-zero linear form on V; this orbit is $\mathrm{PGL}_\mathrm{V}(k)$-equivariantly homeomorphic to $\mathbb{P}(\mathrm{V})(k)$, i.e., to the set of hyperplanes in V.

(3.3) We end this section on the compactified building $\overline{\mathscr{B}}_\delta(\mathrm{PGL}_\mathrm{V},k)$ with a couple of technical results to be used in the next paragraph.

Recall that, for any Banach k-algebra A and any non-Archimedean extension K/k, the formula

$$||f|| = \inf\left\{\max_{i\in\mathrm{I}} |\lambda_i| \cdot ||f_i||;\ \lambda_i \in \mathrm{K},\, f_i \in \mathrm{A} \text{ and } f = \sum_{i\in\mathrm{I}} f_i \otimes \lambda_i\right\}$$

defines a seminorm on the K-algebra $\mathrm{A} \otimes_k \mathrm{K}$ and that $\mathrm{A}\widehat{\otimes}_k\mathrm{K}$ is the Banach K-algebra one gets by completion [**BGR84**, 2.1.7 and 3.4.3]. The following definition is due to Berkovich [**Ber90**, Sect. 5.2].

Definition 3.7. — *Let* X *be a k-analytic space. A point x in* X *is* peaked *if, for any non-Archimedean extension* K/k*, the norm on the Banach* K*-algebra* $\mathscr{H}(x)\widehat{\otimes}_k\mathrm{K}$ *is multiplicative.*

Let x be a peaked point of X*. For any non-Archimedean extension* K/k*, the norm on* $\mathscr{H}(x)\widehat{\otimes}_k\mathrm{K}$ *defines a point in* $\mathscr{M}(\mathscr{H}(x)\widehat{\otimes}_k\mathrm{K})$ *and* $\sigma_\mathrm{K}(x)$ *denotes its image under the canonical map* $\mathscr{M}(\mathscr{H}(x)\widehat{\otimes}_k\mathrm{K}) \to \mathrm{X}\widehat{\otimes}_k\mathrm{K}$.

Remark 3.8. — For a point x in a k-analytic space X, being peaked or not depends only on the completed residue field $\mathscr{H}(x)$.

Lemma 3.9. — *For any point x in $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$, there exists a point y in $\mathbb{A}(\mathrm{V})^{\mathrm{an}}$ lifting x and such that $\mathscr{H}(x) = \mathscr{H}(y)$. In particular, each peaked point x in $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ can be lifted to a peaked point in $\mathbb{A}(\mathrm{V})^{\mathrm{an}}$.*

Proof. This is obvious since the canonical map $\mathbb{A}(\mathrm{V})(\mathrm{K}) - \{0\} \to \mathbb{P}(\mathrm{V})(\mathrm{K})$ is surjective for any field extension K/k. □

Proposition 3.10. — *Let x be a peaked point of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$. For any discretely valued non-Archimedean field* K *extending k, the canonical injection of $\mathscr{X}(\mathrm{V},k)$ into $\mathscr{X}(\mathrm{V},\mathrm{K})$ maps the point $\tau(x)$ to the point $\tau(\sigma_\mathrm{K}(x))$.*

Proof. Consider a peaked point y in $\mathbb{A}(\mathrm{V})^{\mathrm{an}}$ lifting x and denote by $\tau(y)_\mathrm{K}$ the image of $\tau(y)$ under the canonical injection $\mathscr{X}(\mathrm{V},k) \to \mathscr{X}(\mathrm{V},\mathrm{K})$. We want to show: $\tau(y)_\mathrm{K} = \tau(\sigma_\mathrm{K}(y))$.

The point $\sigma_\mathrm{K}(y)$ in $\mathbb{A}(\mathrm{V} \otimes_k \mathrm{K})^{\mathrm{an}}$ is the multiplicative seminorm on $\mathrm{S}^\bullet(\mathrm{V} \otimes_k \mathrm{K}) = (\mathrm{S}^\bullet\mathrm{V}) \otimes_k \mathrm{K}$ defined by

$$|f|(\sigma_\mathrm{K}(y)) = \inf\left\{\max_{i\in\mathrm{I}} |\lambda_i||f_i|(y);\ \lambda_i \in \mathrm{K},\, f_i \in \mathrm{S}^\bullet\mathrm{V} \text{ and } f = \sum_{i\in\mathrm{I}} f_i \otimes \lambda_i\right\}.$$

Hence

$$|f|(\tau(\sigma_{\mathrm{K}}(y))) = \inf\left\{\max_{i\in\mathrm{I}}|\lambda_i||f_i|(y);\ \lambda_i\in\mathrm{K}, f_i\in\mathrm{S}^1\mathrm{V}=\mathrm{V} \text{ and } f=\sum_{i\in\mathrm{I}} f_i\otimes\lambda_i\right\}$$

for any $f\in\mathrm{V}\otimes_k\mathrm{K}$.

Pick a basis $(e_0,\dots,e_d)$ of V diagonalizing $\tau(y)$. Given $f=\sum_{i\in\mathrm{I}} f_i\otimes\lambda_i$ in $\mathrm{V}\otimes_k\mathrm{K}$, we can write $f_i=\sum_{0\leqslant j\leqslant d} a_{ij}e_j$, and

$$\begin{aligned}\max_{i\in\mathrm{I}}|\lambda_i||f_i|(y) &= \max_{i\in\mathrm{I}}|\lambda_i|\max_{0\leqslant j\leqslant d}|a_{ij}||e_j|(y)\\ &= \max_{0\leqslant j\leqslant d}\max_{i\in\mathrm{I}}|\lambda_i||a_{ij}||e_j|(y)\\ &\geqslant \max_{0\leqslant j\leqslant d}\left|\sum_{i\in\mathrm{I}}\lambda_i a_{ij}\right||e_j|(y).\end{aligned}$$

We conclude that

$$\max_{i\in\mathrm{I}}|\lambda_i||f_i|(y)\geqslant|f|(\tau(y)_{\mathrm{K}}),$$

hence $|f|(\tau(y)_{\mathrm{K}})\leqslant|f|(\tau\sigma_{\mathrm{K}}(y))$.

The converse inequality is obvious: for any $f=\sum_{0\leqslant i\leqslant d}a_ie_i$ in $\mathrm{V}\otimes_k\mathrm{K}$,

$$\begin{aligned}|f|(\tau\sigma_{\mathrm{K}}(y))=|f|(\sigma_{\mathrm{K}}(y)) &\leqslant \max_{0\leqslant i\leqslant d}|a_i||e_i|(\sigma_{\mathrm{K}}(y))\\ &\leqslant \max_{0\leqslant i\leqslant d}|a_i||e_i|(y)=|f|(\tau(y)_{\mathrm{K}})|\end{aligned}$$

and we finally get

$$\tau(y)_{\mathrm{K}}=\tau\sigma_{\mathrm{K}}(y).$$

□

4. SATAKE COMPACTIFICATIONS VIA BERKOVICH THEORY

In [**Sat60**], Satake considers a Riemannian symmetric space $\mathrm{S}=\mathrm{G}/\mathrm{K}$ of non-compact type. Using a faithful representation ρ of the real Lie group G in $\mathrm{PSL}(n,\mathbb{C})$, he embeds S in the symmetric space H associated with $\mathrm{PSL}(n,\mathbb{C})$, which can be identified with the space of all positive definite hermitian $n\times n$-matrices of determinant 1. Observing that H has a natural compactification $\overline{\mathrm{H}}$, namely the projectivization of the cone of all positive semidefinite hermitian $n\times n$-matrices, Satake defines the compactification of S associated with ρ as the closure of S in $\overline{\mathrm{H}}$.

In this section and the next one, we present an analogous construction for Bruhat-Tits buildings from two different viewpoints. Let G be a semisimple connected group over a discretely valued non-Archimedean field k. A faithful and absolutely irreducible linear representation $\rho:\mathrm{G}\to\mathrm{GL}_{\mathrm{V}}$ of G in some finite dimensional k-vector space V can be used to embed the building of G in the building of SL_{V}, hence in any compactification of the latter, and we get a compactification of $\mathscr{B}(\mathrm{G},k)$ by taking the closure. The Berkovich compactification of $\mathscr{B}(\mathrm{SL}_{\mathrm{V}},k)$ corresponding to parabolics stabilizing a hyperplane has an elementary description as the space of seminorms up to scaling on V and will be the non-Archimedean analogue of the projective cone of semidefinite hermitian matrices.

The difference between this section and the next one lies in the construction of the map from $\mathscr{B}(\mathrm{G},k)$ to $\mathscr{B}(\mathrm{SL}_{\mathrm{V}},k)$. Whereas functoriality of buildings is a delicate question in general, it is quite remarkable that Berkovich theory allow us to attach very easily and in a completely canonical way a map $\underline{\rho}:\mathscr{B}(\mathrm{G},k)\to\mathscr{B}(\mathrm{PGL}_{\mathrm{V}},k)$ to each absolutely irreducible linear representation $\rho:\mathrm{G}\to\mathrm{GL}_{\mathrm{V}}$. General results of E. Landvogt on functoriality of buildings will be used in the next section.

(4.1) *The map* $\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$. Let G be a semisimple connected k-group and consider a projective representation $\rho : \mathrm{G} \to \mathrm{PGL_V}$, which we assume to be *absolutely irreducible*. We start by showing that the morphism ρ naturally leads to a continuous and $\mathrm{G}(k)$-equivariant map $\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$, whose formation commutes with scalar extension and whose image lies in the building $\mathscr{B}(\mathrm{PGL_V},k)$.

The two main ingredients in the definition of $\underline{\rho}$ are the retraction $\tau : \mathbb{P}(\mathrm{V})^{\mathrm{an}} \to \mathscr{X}(\mathrm{V},k)$, defined in 3.1, and the following well-known fact.

Proposition 4.1. — (i) *For any field extension* K/k *and any Borel subgroup* B *of* $\mathrm{G}\otimes_k \mathrm{K}$, *there exists one and only one* K*-point of* $\mathbb{P}(\mathrm{V})$ *invariant under* B.

(ii) *There exists a unique* k*-morphism* $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ *such that: for any field extension* K/k, *the map* $\widetilde{\rho}_{\mathrm{K}} : \mathrm{Bor}(\mathrm{G})(\mathrm{K}) \to \mathbb{P}(\mathrm{V})(\mathrm{K})$ *sends a Borel subgroup* B *to the unique* K*-point of* $\mathbb{P}(\mathrm{V})$ *invariant under* B.

Proof. We use the following two results:

1. If the field k is algebraically closed, then for each Borel subgroup $\mathrm{B} \in \mathrm{Bor}(\mathrm{G})(k)$ there exists one and only one point in $\mathbb{P}(\mathrm{V})(k)$ invariant under $\mathrm{B}(k)$ [**Che05**, Exposé 15, Proposition 6 and Corollaire 1].
2. If the group G is split over k, then for each Borel subgroup $\mathrm{B} \in \mathrm{Bor}(\mathrm{G})(k)$ there exists at least one point in $\mathbb{P}(\mathrm{V})(k)$ invariant under $\mathrm{B}(k)$ [**Bor91**, Proposition 15.2].

(i) Let K/k be a field extension, pick an algebraic closure K^a of K and consider the separable closure K^s of K in K^a. Given a Borel subgroup B in $\mathrm{G}\otimes_k \mathrm{K}$, assertion 2 provides a K^s-point of $\mathbb{P}(\mathrm{V})$, say x, invariant under the group $\mathrm{B}(\mathrm{K}^s)$. Since the K-scheme B is smooth, the subset $\mathrm{B}(\mathrm{K}^s)$ is dense in B, hence x is invariant under $\mathrm{B}(\mathrm{K}^a)$ and assertion 1 provides uniqueness of this point.

For any $\gamma \in \mathrm{Gal}(\mathrm{K}^s/\mathrm{K})$, the point $\gamma\cdot x$ in $\mathbb{P}(\mathrm{V})(\mathrm{K}^s)$ is invariant under the group $\gamma\mathrm{B}(\mathrm{K}^s) = \mathrm{B}(\mathrm{K}^s)$; uniqueness implies $\gamma\cdot x = x$ and therefore this point belongs to the subset $\mathbb{P}(\mathrm{V})(\mathrm{K})$ of $\mathbb{P}(\mathrm{V})(\mathrm{K}^s)$. We have thus established existence and uniqueness of a $\mathrm{B}(\mathrm{K}^a)$-invariant point in $\mathbb{P}(\mathrm{V})(\mathrm{K})$. We still have to check that this point is fixed by B, i.e., that its image in $\mathbb{P}(\mathrm{V})(\mathrm{S})$ is invariant under the group $\mathrm{B}(\mathrm{S})$ for any K-scheme S.

First step — The functor K-**Sch** $\to$ **Sets**, $\mathrm{S} \mapsto \mathrm{Stab}_{\mathrm{G}(\mathrm{S})}(x)$ *is representable by a closed subgroup, say* Π, *of* G.

As a direct verification shows, the functor K-**Sch** $\to$ **Sets**, $\mathrm{S} \mapsto \mathrm{Stab}_{\mathrm{PGL_V}(\mathrm{S})}(x)$, is represented by a closed and smooth subgroup P_0 of $\mathrm{PGL_V}$. Let Π denote the K-scheme $\mathrm{P}_0 \times_{\mathrm{PGL_V}} \mathrm{G}$. The second projection $\Pi \to \mathrm{G}$ is a closed immersion and Π represents the functor $\mathrm{Stab}_{\mathrm{G}}(x)$ since

$$\Pi(\mathrm{S}) = \{(g,g') \in \mathrm{G}(\mathrm{S}) \times \mathrm{P}_0(\mathrm{S}) \,;\, \rho(g) = g'\} = \mathrm{Stab}_{\mathrm{G}(\mathrm{S})}(x)$$

for any K-scheme S.

Second step — The subgroup B *of* G *is contained in* Π.

Since B is a reduced closed subscheme of G, the inclusion $\mathrm{B}(\mathrm{K}^a) \subset \Pi(\mathrm{K}^a)$ implies the inclusion $\mathrm{B} \subset \Pi$ as subgroups of G and we have thus established that the K-point x of $\mathbb{P}(\mathrm{V})$ is invariant under B. Note also that Π (which may not be smooth) is a generalized parabolic subgroup of G since it contains a Borel subgroup.

(ii) Pick a finite Galois extension k'/k splitting G together with a Borel subgroup B of $\mathrm{G}\otimes_k k'$, and let x be the only k'-point of $\mathbb{P}(\mathrm{V})$ invariant under B. By (i), the map

$$\mathrm{G}(\mathrm{S}) \to \mathbb{P}(\mathrm{V})(\mathrm{S}),\ g \mapsto g\cdot x$$

factors through the canonical projection $\mathrm{G}(\mathrm{S}) \to \mathrm{G}(\mathrm{S})/\mathrm{B}(\mathrm{S})$ for any k'-scheme S. Thanks to the functorial identification $\mathrm{G}(\mathrm{S})/\mathrm{B}(\mathrm{S}) \xrightarrow{\sim} \mathrm{Bor}(\mathrm{G})(\mathrm{S})$, $g\mathrm{B}(\mathrm{S}) \mapsto g(\mathrm{B}\otimes_{k'} \mathrm{S})g^{-1}$ [**SGA3**, Exposé XXVI, Corollaire 5.2] we thus get a morphism of functors $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}\otimes_k k') \to \mathbb{P}(\mathrm{V}\otimes_k k')$ and define therefore a k'-morphism $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}\otimes_k k') \to \mathbb{P}(\mathrm{V}\otimes_k k')$ such that, for any k'-scheme S and any $\mathrm{B}' \in \mathrm{Bor}(\mathrm{G})(\mathrm{S})$,

$\widetilde{\rho}(\mathrm{B}') = g\cdot x$ if $\mathrm{B}' = g\mathrm{B}g^{-1}$, $g \in \mathrm{G}(\mathrm{S})$. In particular, for any field extension K/k, the map $\widetilde{\rho}$ associates with a Borel subgroup $\mathrm{B}' \in \mathrm{Bor}(\mathrm{G})(\mathrm{K})$ the only K-point of $\mathbb{P}(\mathrm{V})$ invariant under B'.

By definition, the k'-morphism

$$\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \otimes_k k' = \mathrm{Bor}(\mathrm{G} \otimes_k k') \to \mathbb{P}(\mathrm{V} \otimes_k k') = \mathbb{P}(\mathrm{V}) \otimes_k k'$$

commutes with the natural action of $\mathrm{Gal}(k'|k)$ and thus $\widetilde{\rho}$ descends to a k-morphism

$$\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$$

satisfying the required condition. □

Proposition 4.2. — *There exists a largest type t of parabolic subgroups of* G *such that the morphism* $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ *factors through the canonical projection* $\mathrm{Bor}(\mathrm{G}) \to \mathrm{Par}_t(\mathrm{G})$. *The so-obtained morphism* $\mathrm{Par}_t(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ *induces a homeomorphism between* $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ *and a closed subspace of* $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$.

Proof. Assume temporarily that the group G is split, pick a Borel subgroup B of G and let $x = \widetilde{\rho}(\mathrm{B})$ be the only k-point of $\mathbb{P}(\mathrm{V})$ invariant under B. If we denote by Π the stabilizer of x in G, then the underlying reduced scheme Π^{red} is the largest parabolic subgroup of G stabilizing x. Indeed, since we have proved above that Π is a closed subgroup containing B, the reduced scheme $(\Pi \otimes_k k^a)^{\mathrm{red}}$ is a smooth closed subgroup of $\mathrm{G} \otimes_k k^a$ containing $\mathrm{B} \otimes_k k^a$, hence a parabolic subgroup of $\mathrm{G} \otimes_k k^a$. As G is split, there exists a unique parabolic subgroup P of G containing B such that $(\Pi \otimes_k k^a)^{\mathrm{red}} = \mathrm{P} \otimes_k k^a$. This identity implies $\mathrm{P} = \Pi^{\mathrm{red}}$, hence Π^{red} is a parabolic subgroup of G stabilizing x. Since each parabolic subgroup Q of G is smooth, Q is a subgroup of Π if it stabilizes x, and therefore Π^{red} contains any parabolic subgroup of G stabilizing x. Note also that the type of Π^{red} does not depend on the choice of B by $\mathrm{G}(k)$-conjugacy of Borel subgroups and equivariance of the map $\widetilde{\rho}$.

The morphism $\widetilde{\rho} : \mathrm{G}/\mathrm{B} \to \mathbb{P}(\mathrm{V})$ induces a map

$$\mathrm{G}/\Pi \hookrightarrow \mathbb{P}(\mathrm{V})$$

which is a monomorphism in the category of k-schemes. Since the image of $\widetilde{\rho}$ is a closed subset of $\mathbb{P}(\mathrm{V})$ by properness of $\mathrm{Bor}(\mathrm{G})$, this map is a closed immersion. Moreover, we have an exact sequence of k-groups

$$e \longrightarrow \Pi/\Pi^{\mathrm{red}} \longrightarrow \mathrm{G}/\Pi^{\mathrm{red}} \xrightarrow{p} \mathrm{G}/\Pi \longrightarrow e$$

and Π/Π^{red} is a finite and connected k-group scheme [**SGA3**, Exposé VIA, 5.6], hence the morphism p is universally injective, i.e., induces an injection between K-points for any extension K of k. Let t denote the rational type of G defined by Π^{red}. Composing p with the morphism $\mathrm{G}/\Pi \to \mathbb{P}(\mathrm{V})$ induced by ρ, we see that $\widetilde{\rho}$ factors through the canonical projection of $\mathrm{Bor}(\mathrm{G})$ onto $\mathrm{Par}_t(\mathrm{G})$. The induced morphism $f : \mathrm{Par}_t(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ is universally injective. At the analytic level, the associated map f^{an} is a continuous injection, hence a homeomorphism onto a closed subset of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ since $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ and $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ are compact.

In general, we pick a finite Galois extension k'/k splitting G and set $\Gamma = \mathrm{Gal}(k'|k)$. For any $\gamma \in \Gamma$, there exists a unique k'-rational type t'_γ such that the morphism ${}^{\gamma}\widetilde{\rho}'_k = \widetilde{\rho} \otimes_k \gamma$ factors through $\mathrm{Par}_{t'_\gamma}(\mathrm{G} \otimes_k k')$. The family $\{t'_\gamma\}_{\gamma\in\Gamma}$ is a Galois orbit, hence defines a type t of parabolic subgroups of G, and the morphism $\widetilde{\rho}$ factors through the canonical projection of $\mathrm{Bor}(\mathrm{G})$ onto $\mathrm{Par}_t(\mathrm{G})$ by Galois descent. □

The above construction associates a well-defined rational type of parabolic subgroups of G with the representation ρ.

Definition 4.3. — *Let ρ be an absolutely irreducible projective representation* $\mathrm{G} \to \mathrm{PGL}_{\mathrm{V}}$. *Its* co-type $t(\check{\rho})$ *is the largest rational type t of* G *such that the canonical morphism* $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ *factors through the projection of* $\mathrm{Bor}(\mathrm{G})$ *onto* $\mathrm{Par}_t(\mathrm{G})$.

Remark 4.4. — This definition is obviously related to the theory of the highest weight: if B is a Borel subgroup of G, then the k-point $\widetilde{\rho}(\mathrm{B})$ of $\mathbb{P}(\mathrm{V})$ is a hyperplane of V invariant under B, hence a line in $\mathrm{V}^\vee$ invariant under B in the contragredient representation $\check{\rho}$. The corresponding character of B is the highest weight of $\check{\rho}$ with respect to B. This observation is the reason why we introduced the *cotype* of the representation ρ; the *type* of ρ should be defined as the cotype of the contragredient representation, i.e., the type of the largest parabolic subgroup stabilizing a highest weight line in V.

Composing the maps

$$\mathscr{B}(\mathrm{G},k) \xrightarrow{\vartheta_\varnothing} \mathrm{Bor}(\mathrm{G})^{\mathrm{an}} \xrightarrow{\widetilde{\rho}} \mathbb{P}(\mathrm{V})^{\mathrm{an}} \xrightarrow{\tau} \mathscr{X}(\mathrm{V},k),$$

we obtain a natural map

$$\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k),$$

canonically associated with the homomorphism $\rho : \mathrm{G} \to \mathrm{PGL}_{\mathrm{V}}$. Since all these maps are continuous and equivariant, so is $\underline{\rho}$.

(4.2) The main properties of $\underline{\rho}$ are easily established. We first consider compatibility with scalar extension.

Proposition 4.5. — *For any discretely valued non-Archimedean field* K *extending* k*, the natural diagram*

$$\begin{array}{ccc} \mathscr{B}(\mathrm{G},\mathrm{K}) & \xrightarrow{\underline{\rho}_{\mathrm{K}}} & \mathscr{X}(\mathrm{V},\mathrm{K}) \\ \uparrow & & \uparrow \\ \mathscr{B}(\mathrm{G},k) & \xrightarrow[\underline{\rho}]{} & \mathscr{X}(\mathrm{V},k) \end{array}$$

is commutative.

The proof of this proposition relies on the following lemma. We recall that, if x is a peaked point of a k-analytic space X and if K/k is a non-Archimedean extension, then $\sigma_{\mathrm{K}}(x)$ denotes the canonical lift of x to $\mathrm{X}\widehat{\otimes}_k\mathrm{K}$ (see Definition 3.7).

Lemma 4.6. — *For any rational type* t *of* G *and any point* x *in* $\overline{\mathscr{B}}_t(\mathrm{G},k)$*, the point* $\vartheta_t(x)$ *of* $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ *is peaked. Moreover, given a non-Archimedean extension* K/k*, the point* $\sigma_{\mathrm{K}}(\vartheta_t(x))$ *of* $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}\widehat{\otimes}_k\mathrm{K}$ *is the image of* x_{K} *under the map*

$$\vartheta_t : \overline{\mathscr{B}}_t(\mathrm{G},\mathrm{K}) \to \mathrm{Par}_t(\mathrm{G}\otimes_k\mathrm{K})^{\mathrm{an}} = \mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}\widehat{\otimes}_k\mathrm{K}.$$

Proof. Let us first consider a finite Galois extension k'/k splitting G and consider a point x' in $\overline{\mathscr{B}}_t(\mathrm{G},k')$. By Proposition 1.13, the point $\vartheta_t(x')$ is contained in some big cell Ω of $\mathrm{Par}_t(\mathrm{G}\otimes_k k')$. Choosing an isomorphism $\mathbb{G}_{\mathrm{a},k'}\tilde{\to}\mathrm{U}_\alpha$ for each root α of $\mathrm{G}\otimes_k k'$ with respect to a maximal split torus T containing $\mathrm{S}\otimes_k k'$ leads to an isomorphism $\mathbb{A}^n_{k'}\tilde{\to}\Omega\otimes_k k'$. Then the point $\vartheta_t(x')$ corresponds to a seminorm on the algebra $k'[\xi_1,\dots,\xi_n]$ of the form

$$\sum_\nu a_\nu \xi^\nu \mapsto \max_\nu |a_\nu| \prod_{i=1}^n c_i^{\nu_i},$$

where $c_1,\dots,c_n$ are non-negative real numbers, not all equal to zero (with the convention $0^0 = 1$). Such a seminorm defines a peaked point in $\mathbb{A}^{n,\mathrm{an}}_{k'}$ [**Ber90**, Sect. 5.2] and the point $\vartheta_t(x')$ is therefore peaked.

In general, pick a point x in $\overline{\mathscr{B}}_t(\mathrm{G},k)$ and let $x_{k'}$ denote its image in $\mathscr{B}(\mathrm{G},k')$, where k'/k is a finite Galois extension splitting G. We consider the completed residue field $\mathscr{H}(\vartheta_t(x))$ of $\vartheta_t(x)$. The point $\vartheta_t(x_{k'})$ induces a norm on the k'-Banach algebra $\mathscr{H}(\vartheta_t(x))\otimes_k k'$ with respect to which the descent datum is an isometry (note that $\mathscr{H}(\vartheta_t(x))\otimes_k k'$ is finite extension of k'). Since the point $\vartheta_t(x_{k'})$ is

peaked, this norm is universally multiplicative. By [**RTW09**, Lemma A.10], it follows that the norm induced on $\mathscr{H}(\vartheta_t(x))$ is also universally multiplicative, hence the point $\vartheta_t(x)$ is peaked.

In order to prove the second assertion, consider a point x in $\overline{\mathscr{B}}_t(\mathrm{G},k)$ and let K/k be a non-Archimedean extension. Since the point $\vartheta_t(x)$ is peaked, the Banach norm on the K-Banach algebra $\mathscr{H}(\vartheta_t(x))\widehat{\otimes}_k\mathrm{K}$ coming from the absolute value of $\mathscr{H}(\vartheta_t(x))$ is multiplicative. On the other hand, the point $\vartheta_t(x_\mathrm{K})$ also defines a multiplicative norm on this K-Banach algebra. Two such norms necessarily coincide, hence $\sigma_\mathrm{K}(\vartheta_t(x)) = \vartheta_t(x_\mathrm{K})$. □

Proof of Proposition 4.5. Let K be a discretely valued non-Archimedean field extending k. Denoting by t the cotype of the representation ρ, the morphism $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ factors through the canonical projection $\mathrm{Bor}(\mathrm{G}) \to \mathrm{Par}_t(\mathrm{G})$ and leads to a homeomorphism between $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ and a closed subset of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ (Proposition 4.2). Pick a point x in $\mathscr{B}(\mathrm{G},k)$. The point $\widetilde{\rho}(\vartheta_t(x))$ of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ is peaked since $\mathscr{H}(\widetilde{\rho}\vartheta_t(x)) = \mathscr{H}(\vartheta_t(x))$ and $\vartheta_t(x)$ is a peaked point of $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ (Lemma 4.6). Moreover, we have the identities

$$\sigma_\mathrm{K}\widetilde{\rho}\,\vartheta_t(x) = \widetilde{\rho_\mathrm{K}}\sigma_\mathrm{K}\vartheta_t(x) = \widetilde{\rho_\mathrm{K}}\vartheta_t(x).$$

The conclusion finally follows from Proposition 3.10: the points $\underline{\rho}(x) = \tau\widetilde{\rho}\,\vartheta_t(x)$ and $\underline{\rho_\mathrm{K}}(x) = \tau\widetilde{\rho_\mathrm{K}}\vartheta_t(x) = \tau\sigma_\mathrm{K}\widetilde{\rho}\,\vartheta_t(x)$ coincide in $\mathscr{X}(\mathrm{V},\mathrm{K})$. □

Proposition 4.7. — *The image of the map $\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$ is contained in the open stratum $\mathscr{B}(\mathrm{PGL}_\mathrm{V},k)$ of $\mathscr{X}(\mathrm{V},k)$.*

Proof. Assume that there exists a point x in $\mathscr{B}(\mathrm{G},k)$ whose image under the map $\underline{\rho}$ is not contained the open stratum $\mathscr{B}(\mathrm{PGL}_\mathrm{V},k)$ of $\mathscr{X}(\mathrm{V},k)$. Under this hypothesis, the point $\underline{\rho}(x) = \tau\widetilde{\rho}\,\vartheta_\varnothing(x)$ lies in $\mathscr{X}(\mathrm{V},k)\cap\mathbb{P}(\mathrm{V}/\mathrm{W})^{\mathrm{an}}$ for some non trivial linear subspace W in V, hence $\widetilde{\rho}\,\vartheta_\varnothing(x) \in \mathbb{P}(\mathrm{V}/\mathrm{W})^{\mathrm{an}}$. Now consider the following diagram

$$\begin{array}{ccc} \mathrm{Bor}(\mathrm{G})^{\mathrm{an}} & \xrightarrow{\widetilde{\rho}} & \mathbb{P}(\mathrm{V})^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathrm{Bor}(\mathrm{G}) & \xrightarrow[\widetilde{\rho}]{} & \mathbb{P}(\mathrm{V}) \end{array}$$

in which the vertical arrows are the maps sending a point z of X^{an}, seen as a multiplicative seminorm on the algebra $\mathscr{O}_\mathrm{X}(\mathrm{U})$ of some open affine subset U of X, to the point of the scheme X defined by the prime ideal $\ker(z) \in \mathrm{Spec}(\mathscr{O}_\mathrm{X}(\mathrm{U}))$ (where $\mathrm{X} = \mathrm{Bor}(\mathrm{G})$, or $\mathrm{X} = \mathbb{P}(\mathrm{V})$). The point x ($\widetilde{\rho}(x)$, respectively) is mapped to the generic point of $\mathrm{Bor}(\mathrm{G})$ (to the generic point of $\mathbb{P}(\mathrm{V}/\mathrm{W})$, respectively). Since the diagram above is commutative, it follows that the morphism $\widetilde{\rho}$ maps the generic point of $\mathrm{Bor}(\mathrm{G})$ to the generic point of $\mathbb{P}(\mathrm{V}/\mathrm{W})$, hence maps $\mathrm{Bor}(\mathrm{G})$ into the strict linear subspace $\mathbb{P}(\mathrm{V}/\mathrm{W})$ of $\mathbb{P}(\mathrm{V})$. Hence it would follow that ρ maps G into the nontrivial parabolic subgroup of PGL_V stabilizing $\mathbb{P}(\mathrm{V}/\mathrm{W})$, thereby contradicting the irreducibility of ρ. □

(4.3) We now state and prove the main result of this section.

Theorem 4.8. — *Let k be a discretely valued non-Archimedean field and* G *a semisimple connected k-group. We consider a finite-dimensional k-vector space* V *and an absolutely irreducible projective representation $\rho : \mathrm{G} \to \mathrm{PGL}_\mathrm{V}$.*

(i) *The map $\underline{\rho} : \mathscr{B}(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$ extends continuously to the compactification $\mathscr{B}(\mathrm{G},k) \hookrightarrow \overline{\mathscr{B}}_{t(\check{\rho})}(\mathrm{G},k)$.*

(ii) *The induced map is an injection of $\overline{\mathscr{B}}_{t(\check{\rho})}(\mathrm{G},k)$ into $\mathscr{X}(\mathrm{V},k)$.*

(iii) *If the field k is locally compact, the map $\underline{\rho}$ extends to a homeomorphism between $\overline{\mathscr{B}}_{t(\check{\rho})}(\mathrm{G},k)$ and the closure of $\underline{\rho}(\mathscr{B}(\mathrm{G},k))$ in $\mathscr{X}(\mathrm{V},k)$.*

Proof. Set $t = t(\check{\rho})$.

(i) The morphism $\widetilde{\rho} : \mathrm{Bor}(\mathrm{G}) \to \mathbb{P}(\mathrm{V})$ factors through the canonical projection $\pi_t : \mathrm{Bor}(\mathrm{G}) \to \mathrm{Par}_t(\mathrm{G})$ and leads to a homeomorphism between $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ and a closed subset of $\mathbb{P}(\mathrm{V})^{\mathrm{an}}$ (Proposition 4.2). The diagram

$$\begin{array}{ccccccc} \mathscr{B}(\mathrm{G},k) & \xrightarrow{\vartheta_\varnothing} & \mathrm{Bor}(\mathrm{G})^{\mathrm{an}} & \xrightarrow{\widetilde{\rho}} & \mathbb{P}(\mathrm{V})^{\mathrm{an}} & \xrightarrow{\tau} & \mathscr{X}(\mathrm{V},k) \\ & \searrow^{\vartheta_t} & \downarrow{\scriptstyle \pi_t} & \nearrow_{\widetilde{\rho}} & & & \\ & & \mathrm{Par}_t(\mathrm{G})^{\mathrm{an}} & & & & \end{array}$$

is commutative (use [**RTW09**, section 4.2] for the left-hand side triangle) and hence allows us to write the map $\underline{\rho}$ as the composition $\tau\widetilde{\rho}\vartheta_t$. For any maximal split torus S in G, the restriction of $\underline{\rho}$ to the apartment $\mathrm{A}(\mathrm{S},k)$ extends continuously to its closure $\overline{\mathrm{A}}_t(\mathrm{S},k)$ in $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$. Since the image of $\overline{\mathscr{B}}_t(\mathrm{G},k)$ into $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ is the union of these closures when S runs over all maximal split tori of G, the maps $\underline{\rho}$ extends to $\overline{\mathscr{B}}_t(\mathrm{G},k)$. This extension is continuous, for it is $\mathrm{G}(k)$-equivariant and its restriction to $\overline{\mathrm{A}}_t(\mathrm{S},k)$ is continuous.

(ii) Let us now prove that the map $\overline{\mathscr{B}}_t(\mathrm{G},k) \to \mathscr{X}(\mathrm{V},k)$ extending $\underline{\rho}$, for which we keep the notation $\underline{\rho}$, is injective. The fact that compatibility of $\underline{\rho}$ with scalar extension is proved only for discretely valued non-Archimedean extensions of k in Proposition 4.5 is a slight difficulty.

Given two points $x, y \in \overline{\mathscr{B}}_t(\mathrm{G},k)$ with $\underline{\rho}(x) = \underline{\rho}(y)$, we will show that $\mathrm{G}_x(k^{\mathrm{a}}) = \mathrm{G}_y(k^{\mathrm{a}})$, where $\mathrm{G}_x = \mathrm{Stab}^t_{\mathrm{G}}(x)$ and $\mathrm{G}_y = \mathrm{Stab}^t_{\mathrm{G}}(y)$. Since the field k is discretely valued, it follows from its description as a disjoint union of buildings (cf. Theorem 1.4) that the compactified building $\overline{\mathscr{B}}_t(\mathrm{G},k)$ carries a (poly-)simplicial decomposition and, by application of Bruhat-Tits theory to each stratum, the fixed-point set of $\mathrm{Stab}^t_{\mathrm{G}}(x)(k)$ is precisely the facet of $\overline{\mathscr{B}}_t(\mathrm{G},k)$ whose interior contains the point x. Now, since two distinct points of $\overline{\mathscr{B}}_t(\mathrm{G},k)$ belong to disjoint facets of $\overline{\mathscr{B}}_t(\mathrm{G},k')$ for a large enough finite extension k'/k, the equality $\mathrm{G}_x(k^{\mathrm{a}}) = \mathrm{G}_y(k^{\mathrm{a}})$ implies $x = y$.

We pick a point x in $\overline{\mathscr{B}}_t(\mathrm{G},k)$ and set $\mathrm{G}_{\underline{\rho}(x)} = \rho^{-1}\left(\mathrm{Stab}^{\delta}_{\mathrm{PGL_V}}(\underline{\rho}(x))\right)^{\mathrm{red}}$. This is an analytic subgroup of G^{an}, and

$$\mathrm{G}_{\underline{\rho}(x)}(\mathrm{K}) = \{g \in \mathrm{G}(\mathrm{K}) \; ; \; \rho(g)\underline{\rho}(x) = \underline{\rho}(x)\}$$

for any non-Archimedean extension K/k. Given any finite extension k'/k, it follows from Proposition 4.5 that $\mathrm{G}_{\underline{\rho}(x)}(k')$ contains $\mathrm{G}_x(k')$. We have therefore $\mathrm{G}_x(k^{\mathrm{a}}) \subset \mathrm{G}_{\underline{\rho}(x)}(k^{\mathrm{a}})$, and we will now prove that equality holds. Notice that, if the point x is rational (i.e., if it becomes a vertex over some finite extension of k), then the inclusion $\mathrm{G}_x(k^{\mathrm{a}}) \subset \mathrm{G}_{\underline{\rho}(x)}(k^{\mathrm{a}})$ implies $\mathrm{G}_x \subset \mathrm{G}_{\underline{\rho}(x)}$ by density.

Notation — The point x belongs to a stratum S. Let $\mathrm{P} = \mathrm{Stab}_{\mathrm{G}}(\mathrm{S})$ denote the corresponding t-relevant parabolic subgroup of G and let $\mathrm{R} = \mathrm{R}_t(\mathrm{P})$ denote the largest connected, smooth and normal subgroup of G acting trivially on $\mathrm{Osc}_t(\mathrm{P})$. Similarly, the point $\underline{\rho}(x)$ belongs to a stratum Σ of $\mathscr{X}(\mathrm{V},k)$; we set $\Pi = \mathrm{Stab}_{\mathrm{PGL_V}}(\Sigma)$ and we let $\mathrm{R}_\delta(\Pi)$ denote the largest connected, smooth and normal subgroup of Π acting trivially on Σ. Up to replacing k by a finite extension, we may assume that the reduced subschemes $\mathrm{P}' = \rho^{-1}(\Pi)^{\mathrm{red}}$ and $\mathrm{R}'' = \rho^{-1}(\mathrm{R}_\delta(\Pi))^{\mathrm{red}}$ are smooth subgroups of G. Note that R'' is connected and invariant in P'.

First step — The group $\mathrm{G}_x(k)$ is Zariski-dense in P (Theorem 1.12) and ρ maps $\mathrm{G}_{\underline{\rho}(x)}(k)$ into $\Pi(k)$. Since P is reduced, the inclusion $\mathrm{G}_x(k) \subset \mathrm{G}_{\underline{\rho}(x)}(k)$ implies that ρ maps P into Π and therefore $\mathrm{P}' = \rho^{-1}(\Pi)^{\mathrm{red}}$ is a parabolic subgroup of G containing P.

This parabolic subgroup P' defines a stratum S' in $\overline{\mathscr{B}}_t(\mathrm{G},k)$, the only one it stabilizes. We have $\mathrm{S} \subset \overline{\mathrm{S}'}$ since $\mathrm{P} \subset \mathrm{P}'$, and $\mathrm{S} = \mathrm{S}'$ if and only if $\mathrm{P} = \mathrm{P}'$ for P is t-relevant. In order to establish the last identity, we let $\mathrm{R}' = \mathrm{R}_t(\mathrm{P}')$ denote the largest smooth connected and normal subgroup of P' acting trivially on $\mathrm{Osc}_t(\mathrm{P}')$.

Second step — We now prove that the parabolic subgroups P and P′ coincide.

Since $P' = \rho^{-1}(\Pi)^{red}$, the morphism $\widetilde{\rho}$ maps the closed subscheme $Osc_t(P')$ of $Par_t(G)$ to the closed subscheme $Osc_\delta(\Pi)$ of $\mathbb{P}(V)$. By construction, $\widetilde{\rho}$ is universally injective (i.e., purely inseparable), hence the induced map $Osc_t(P')(K) \to Osc_\delta(\Pi)(K)$ is injective for any extension K of k. It follows that any element g of $R''(K)$ acts trivially on $Osc_t(P')(K)$, which implies that the action of R'' on the reduced scheme $Osc_t(P')$ is itself trivial. As the subgroup R'' is smooth, connected and normal in P′, we deduce that R'' is contained in R' by maximality of the latter. On the other hand, R'^{an} is trivially contained in R^{an}, hence in $G_{\underline{\rho(x)}}$, since any element acting trivially on S′ fixes $\overline{S'}$ pointwise.

We consider now the quotient group $H = P'/R'$, which is semisimple and satisfies $S' = \mathscr{B}(H,k)$. Thanks to the inclusion $R'' \subset R'$, this group is also a quotient of P'/R''. Since $P' = \rho^{-1}(\Pi)^{red}$ and $R'' = \rho^{-1}(R_\delta(\Pi))^{red}$, we get a canonical morphism

$$p:\ P'/R'' \hookrightarrow \rho^{-1}(\Pi)/\rho^{-1}(R_\delta(\Pi)) \hookrightarrow \Gamma = \Pi/R_\delta(\Pi)$$

which is finite. By construction, we have $R''^{an} \subset G_{\underline{\rho(x)}} \subset P'^{an}$ and $G_{\underline{\rho(x)}}/R''^{an} = p^{-1}(\Gamma_{\underline{\rho(x)}})^{red}$, hence $G_{\underline{\rho(x)}}/R''^{an}$ is bounded in $(P'/R'')^{an}$ for p is finite. It follows that $G_{\underline{\rho(x)}}/R'^{an}$ is a bounded in H^{an}.

Since $G_x(k^a) \subset G_{\underline{\rho(x)}}(k^a)$, the discussion above shows that the stabilizer $(G_x/R')(k^a)$ of x in $H(k^a)$ is bounded. By Remark 1.11, this amounts to saying that x belongs to the open stratum of $\overline{S} = \overline{\mathscr{B}}_t(H,k)$, hence $S' = S$ and $P' = P$.

Third step — We have just proved that the subgroup $G_{\underline{\rho(x)}}(k^a)$ of $G(k^a)$ is contained in the parabolic subgroup P and has bounded image in the quotient group $H = P/R$. The inclusion $G_x(k^a) \subset G_{\underline{\rho(x)}}(k^a)$ implies $G_x(k^a) = G_{\underline{\rho(x)}}(k^a)$ since $(G_x/R^{an})(k^a) = G_x(k^a)/R(k^a)$ is a maximal bounded subgroup of $H(k^a)$.

(iii) If the field k is locally compact, the continuous extension of $\underline{\rho} : \mathscr{B}(G,k) \to \mathscr{X}(V,k)$ to $\overline{\mathscr{B}}_{t(\check{\rho})}(G,k)$ is continuous injection between two locally compact spaces, hence is a homeomorphism on its image. □

(4.3) We end this section by establishing a natural and expected property of $\underline{\rho}$.

Proposition 4.9. — *For any maximal split torus* S *of* G, *there exists a maximal split torus* T *of* PGL_V *containing* $\rho(S)$ *and such that* $\underline{\rho}$ *maps* $A(S,k)$ *into* $A(T,k)$.

Proof. For any finite extension k'/k, we normalize the metrics so that the canonical embeddings $\mathscr{B}(G,k) \hookrightarrow \mathscr{B}(G,k')$ and $\mathscr{B}(PGL_V,k) \hookrightarrow \mathscr{B}(PGL_V,k')$ are isometric.

Given a maximal split torus S of G, our first goal is to find an apartment A′ of $\mathscr{B}(PGL_V,k)$ containing the image of $A(S,k)$.

Let T be a maximal split torus of PGL_V containing $\rho(S)$ and let x be a point in $A(S,k)$. For any $s \in S(k)$, we have $\underline{\rho}(s\cdot x) = \rho(s)\cdot\underline{\rho}(x)$ and $\rho(s)\cdot A(T,k) = A(T,k)$, hence

$$\operatorname{dist}\left(\underline{\rho}(s\cdot x), A(T,k)\right) = \operatorname{dist}\left(\underline{\rho}(x), A(T,k)\right).$$

More generally, we have

$$\operatorname{dist}\left(\underline{\rho}(s\cdot x), A(T,k')\right) = \operatorname{dist}\left(\underline{\rho}(x), A(T,k')\right)$$

for any finite extension k'/k and any $s \in S(k')$. Since the points of $A(S,k)$ belonging to the orbit of x under $S(k')$ for some finite extension k'/k are dense (in $A(S,k)$), it follows that $\operatorname{dist}\left(\underline{\rho}(z), A(T,k)\right)$ is independent of $z \in A(S,k)$. Now, the existence of a maximal split torus T′ of PGL_V such that $\rho(S) \subset T'$ and $\underline{\rho}(x) \in A(T',k)$, hence such that $\underline{\rho}(A(S,k)) \subset A(T',k)$, follows immediately from the next two facts:

1. the set of distances of $\underline{\rho}(x)$ to apartments in $\mathscr{B}(PGL_V,k)$ is discrete;

2. given a maximal split torus T of $\mathrm{PGL_V}$ such that $\rho(\mathrm{S}) \subset \mathrm{T}$ and $\underline{\rho}(x) \notin \mathrm{A}(\mathrm{T},k)$, there exists a maximal split torus T' of $\mathrm{PGL_V}$ satisfying $\rho(\mathrm{S}) \subset \mathrm{T}'$ and

$$\mathrm{dist}\left(\underline{\rho}(x), \mathrm{A}(\mathrm{T}',k)\right) < \mathrm{dist}\left(\underline{\rho}(x), \mathrm{A}(\mathrm{T},k)\right).$$

The first assertion follows easily from the (poly-)simplicial structure on $\mathscr{B}(\mathrm{PGL_V},k)$, hence from the fact that the field k is discretely valued. Let us then prove the second assertion.

For any point $z \in \mathrm{A}(\mathrm{S},k)$, let $p(z)$ denote the unique point of $\mathrm{A}(\mathrm{T},k)$ satisfying

$$\mathrm{dist}\left(\underline{\rho}(z), p(z)\right) = \mathrm{dist}\left(\underline{\rho}(z), \mathrm{A}(\mathrm{T},k)\right)$$

and observe that the image of the map

$$p : \mathrm{A}(\mathrm{S},k) \to \mathrm{A}(\mathrm{T},k), \;\; z \mapsto p(z)$$

is an affine subspace under the image of $\Lambda(\mathrm{S})$ in $\Lambda(\mathrm{T})$.

We now use the (poly-)simplicial structure on $\mathrm{A}(\mathrm{T},k)$. Suppose that there exists a point $z \in \mathrm{A}(\mathrm{S},k)$ such that p(z) belongs to the interior of an alcove c. Any path in $\mathscr{B}(\mathrm{PGL_V},k)$ from $p(z)$ to a point lying outside $\mathrm{A}(\mathrm{T},k)$ contains an initial segment $[p(z),z']$ with $z' \in \partial c$ and $[p(z),z'[\subset c$. Applied to the geodesic path $[p(z),\underline{\rho}(z)]$, this observation leads to a contradiction if $\mathrm{dist}\left(\underline{\rho}(z), \mathrm{A}(\mathrm{T},k)\right) > 0$, since then

$$\mathrm{dist}\left(\underline{\rho}(z), \mathrm{A}(\mathrm{T},k)\right) = \mathrm{dist}\left(\underline{\rho}(z), p(z)\right) < \mathrm{dist}\left(\underline{\rho}(z), z'\right) \leqslant \mathrm{dist}\left(\underline{\rho}(z), \mathrm{A}(\mathrm{T},k)\right).$$

Therefore, since $\underline{\rho}(x) \notin \mathrm{A}(\mathrm{T},k)$, the affine subspace $p(\mathrm{A}(\mathrm{S},k))$ of $\mathrm{A}(\mathrm{T},k)$ is contained in some root hyperplane $\mathrm{H}_{\alpha,r} = \{\alpha = r\}$, where $\alpha \in \mathrm{X}^*(\mathrm{T})$ is a root whose restriction to S is trivial: $\alpha_{|\mathrm{S}} = 1$, and $r \in |k^\times|$. By folding $\mathrm{A}(\mathrm{T},k)$ along $\mathrm{H}_{\alpha,r}$, we will obtain a new apartment of $\mathscr{B}(\mathrm{PGL_V},k)$ which is closer to $\underline{\rho}(\mathrm{A}(\mathrm{S},k))$.

Let $x_0 = \underline{\rho}(x), x_1, \ldots, x_n = p(x)$ denote the successive vertices of the simplicial decomposition of $[\underline{\rho}(x), p(x)]$ induced by the (poly-)simplicial structure of $\mathscr{B}(\mathrm{PGL_V},k)$. There exists an element u of $\mathrm{U}_\alpha(k)_r$ satisfying the following two conditions:

(a) $\mathrm{A}(\mathrm{T},k) \cap u \cdot \mathrm{A}(\mathrm{T},k)$ is the half-apartment $\{\alpha \leqslant r\}$;
(b) $u \cdot \mathrm{A}(\mathrm{T},k) = \mathrm{A}(u\mathrm{T}u^{-1},k)$ contains $[x_{n-1},x_n]$.

Since $\alpha_{|\mathrm{S}} = 1$, we have $sus^{-1} = u$ for any $s \in \mathrm{S}(k)$ and thus $\rho(\mathrm{S}(k'))$ stabilizes the apartment $\mathrm{A}(u\mathrm{T}u^{-1},k')$ for any finite extension k'/k. Setting $\mathrm{N} = \mathrm{Norm}_{\mathrm{PGL_V}}(u\mathrm{T}u^{-1})$, the stabilizer of $\mathrm{A}(u\mathrm{T}u^{-1},k')$ in $\mathrm{PGL_V}(k')$ is the group $\mathrm{N}(k')$, hence $\rho(\mathrm{S}(k')) \subset \mathrm{N}(k')$ for any finite extension k'/k and thus $\rho(\mathrm{S}) \subset \mathrm{N}$ since both S and N are reduced k-groups. By connectedness, it follows that S is contained in $\mathrm{N}^\circ = u\mathrm{T}u^{-1}$.

We have

$$\mathrm{dist}\left(\underline{\rho}(x), \mathrm{A}(u\mathrm{T}u^{-1},k)\right) \leqslant \mathrm{dist}(\underline{\rho}(x), x_{n-1}) = \mathrm{dist}(\underline{\rho}(x), x_n) - \mathrm{dist}(x_{n-1}, x_n) < \mathrm{dist}\left(\underline{\rho}(x), \mathrm{A}(\mathrm{T},k)\right)$$

since $x_{n-1} \neq x_n$. This concludes the proof of assertion 2 above.

We have just proved that there exists a maximal split torus T' of $\mathrm{PGL_V}$ such that $\rho(\mathrm{S}) \subset \mathrm{T}'$ and $\underline{\rho}(\mathrm{A}(\mathrm{S},k)) \subset \mathrm{A}(\mathrm{T}',k)$. Thanks to compatibility of $\underline{\rho}$ with finite field extensions, the inclusion $\underline{\rho}(\mathrm{A}(\mathrm{S},k)) \subset \mathrm{A}(\mathrm{T}',k)$ holds more generally after any such extension. As before, it follows that $\rho(\mathrm{S})$ is contained in $\mathrm{T}' = \mathrm{Norm}_{\mathrm{PGL_V}}(\mathrm{T}')^\circ$ and this completes the proof. □

Remark 4.10. — Given two semisimple connected k-groups G, H and a homomorphism $f : \mathrm{G} \to \mathrm{H}$, the above proof applies more generally to any continuous and $\mathrm{G}(k)$-equivariant map $\mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{H},k)$ which is compatible with finite extensions of k: the apartment of any maximal split torus S of G is mapped to the apartment of a maximal split torus of H containing $f(\mathrm{S})$.

Functoriality of buildings with respect to group homomorphisms has been studied by Landvogt in [**Lan00**]. Given a complete discretely valued field k with perfect residue field and two semisimple connected k-groups G and H, Landvogt proved that each homomorphism $f : \mathrm{G} \to \mathrm{H}$ gives rise to a non-empty set of $\mathrm{G}(k)$-equivariant and continuous maps $f_* : \mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{H},k)$. By construction, each such map is *toral*, i.e., maps the apartment of a maximal split torus S of G to the apartment of a maximal split torus of H containing $f(\mathrm{S})$. In the special case where $\mathrm{H} = \mathrm{PGL_V}$ and f is an absolutely irreducible representation, the map $\underline{f}$ introduced in this section is an instance of Landvogt's maps.

The canonical nature of the map $\underline{f}$ raises two obvious questions: is the set of Landvogt's maps reduced to an element when f is an absolutely irreducible representation? If no, is there a way to single out $\underline{f}$ without using Berkovich geometry?

5. Satake compactifications via Landvogt's functoriality

In this last section, we present another approach to Satake compactifications using Landvogt's results on functoriality of Bruhat-Tits buildings. As before, G is a connected, semisimple group over a non-Archimedean local field k. We fix a faithful, absolutely irreducible representation $\rho : \mathrm{G} \to \mathrm{GL_V}$ for some finite-dimensional k-vector space V. Using results from [**Lan00**], the representation ρ defines a continuous, $\mathrm{G}(k)$-equivariant embedding $\rho_* : \mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{SL_V},k)$.

As in the previous section, we want to use one fixed compactification of $\mathscr{B}(\mathrm{SL_V},k)$ on the right-hand side and take the closure of the image of $\mathscr{B}(\mathrm{G},k)$ to retrieve $\overline{\mathscr{B}}(\mathrm{G},k)_\rho$. For functoriality reasons, the natural candidate for this compactification of $\mathscr{B}(\mathrm{SL_V},k)$ is $\overline{\mathscr{B}}(\mathrm{SL_V},k)_{\mathrm{id}}$ for the identical representation $\mathrm{id} : \mathrm{SL_V} \to \mathrm{GL_V}$. According to Theorem 2.1, $\overline{\mathscr{B}}(\mathrm{SL_V},k)_{\mathrm{id}} = \overline{\mathscr{B}}_\pi(\mathrm{SL_V},k)$, where π is the type of parabolics stabilizing a line in V. This space was studied in [**Wer01**] and is canonically isomorphic to $\overline{\mathscr{B}}_\delta(\mathrm{SL_{V^\vee}},k)$, where $\mathrm{V}^\vee$ denotes the dual vector space. It can be identified with the union of all Bruhat-Tits buildings $\mathscr{B}(\mathrm{SL_{V'}},k)$, where V′ runs through the linear subspaces of V. Its points can be described as seminorms on $\mathrm{V}^\vee$ up to scaling and vertices correspond bijectively to the homothety classes of free k°-submodules (of arbitrary rank) in V.

In the following, we let τ denote the unique k-rational type such that $\overline{\mathscr{B}}(\mathrm{G},k)_\rho \cong \overline{\mathscr{B}}_\tau(\mathrm{G},k)$, whose existence was established in section 2. It will eventually turn out that we can replace τ by the (non necessarily k-rational) type $t(\rho)$ naturally associated with ρ.

(5.1) We recall some results of [**Lan00**], applied to the representation $\rho : \mathrm{G} \to \mathrm{GL_V}$. Since G is semisimple, it is equal to its derived group. Hence ρ comes from a representation $\rho : \mathrm{G} \to \mathrm{SL_V}$, for which we use the same notation.

Let S be a maximal split torus in G with normalizer N, and let $\mathrm{A}(\mathrm{S},k)$ denote the corresponding apartment in $\mathscr{B}(\mathrm{G},k)$. Choose a special vertex o in $\mathrm{A}(\mathrm{S},k)$. By [**Lan00**], there exists a maximal split torus T in $\mathrm{SL_V}$ containing $\rho(\mathrm{S})$, and there exists a point o' in the apartment $\mathrm{A}(\mathrm{T},k)$ of T such that the following properties hold:

1. There is a unique affine map $i : \mathrm{A}(\mathrm{S},k) \to \mathrm{A}(\mathrm{T},k)$ such that $i(o) = o'$. Its linear part is induced by $\rho : \mathrm{S} \to \mathrm{T}$.
2. The map i satisfies $\rho(\mathrm{P}_x) \subset \mathrm{P}'_{i(x)}$ for all $x \in \mathrm{A}(\mathrm{S},k)$, where P_x denotes the stabilizer of the point x with respect to the $\mathrm{G}(k)$-action on $\mathscr{B}(\mathrm{G},k)$, and $\mathrm{P}'_{i(x)}$ denotes the stabilizer of the point $i(x)$ with respect to the $\mathrm{SL_V}(k)$-action on $\mathscr{B}(\mathrm{SL_V},k)$.
3. The map $\rho_* : \mathrm{A}(\mathrm{S},k) \to \mathrm{A}(\mathrm{T},k) \to \mathscr{B}(\mathrm{SL_V},k)$ defined by composing i with the natural embedding of the apartment $\mathrm{A}(\mathrm{T},k)$ in the building $\mathscr{B}(\mathrm{SL_V},k)$ is $\mathrm{N}(k)$-equivariant, i.e., for all $x \in \mathrm{A}(\mathrm{S},k)$ and $n \in \mathrm{N}(k)$ we have $\rho_*(nx) = \rho(n)\rho_*(x)$.

These properties imply that $\rho_* : \mathrm{A}(\mathrm{S},k) \to \mathscr{B}(\mathrm{SL_V},k)$ can be continued to a map $\rho_* : \mathscr{B}(\mathrm{G},k) \to \mathscr{B}(\mathrm{SL_V},k)$, which is continuous and $\mathrm{G}(k)$-equivariant. By [**Lan00**, 2.2.9], ρ_* is injective and isometrical, if the metric on $\mathscr{B}(\mathrm{G},k)$ is normalized correctly.

We want to show that ρ_* can be extended to a map $\rho_* : \overline{\mathscr{B}}(\mathrm{G},k)_\rho \cong \overline{\mathscr{B}}_\tau(\mathrm{G},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL_V},k)$. Besides, we prove that this map of compactified buildings identifies $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ as a topological $\mathrm{G}(k)$-space with the closure of $\rho_*(\mathscr{B}(\mathrm{G},k))$ in $\overline{\mathscr{B}}_\pi(\mathrm{SL_V},k)$.

(5.2) Let us first look at compactified apartments in $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ and $\overline{\mathscr{B}}_\pi(\mathrm{SL_V},k)$.

Let $(e_0,\dots,e_d)$ be a basis of V consisting of eigenvectors of T and denote by $\chi_0,\dots,\chi_d$ the corresponding characters of T. The map

$$\Lambda(\mathrm{T}) \to (\mathbb{R}_{>0})^{d+1},\ \ u \mapsto (\langle u,\chi_i\rangle)_{0\leqslant i\leqslant d}$$

identifies $\Lambda(\mathrm{T})$ with the subset of $(\mathbb{R}_{>0})^{d+1}$ consisting of vectors $(r_0,\dots,r_d)$ satisfying $r_0\dots r_d = 1$. The fan on $\Lambda(\mathrm{T})$ defining the compactification $\overline{\mathrm{A}}_\pi(\mathrm{T},k)$ of $\mathrm{A}(\mathrm{T},k)$ consists of all faces of the cones $\mathrm{C}_0,\dots,\mathrm{C}_d$, where

$$\mathrm{C}_i = \{(r_0,\dots,r_d) \in (\mathbb{R}_{>0})^{d+1}\ ;\ r_0\cdot\ldots\cdot r_d = 1 \ \text{ and } \ r_i \geqslant r_j, \ \text{ for all j}\}.$$

The weights of the representation ρ with respect to the torus S are the images of $\chi_0,\dots,\chi_d$ under the projection $\mathrm{X}^*(\mathrm{T}) \to \mathrm{X}^*(\mathrm{S})$ deduced from the morphism $\rho : \mathrm{S} \to \mathrm{T}$, i.e., the restrictions of $\chi_0,\dots,\chi_d$ to S. Setting $\lambda_i = (\chi_i)_{|\mathrm{S}}$ for all $i \in \{0,\dots,d\}$ and identifying as above $\Lambda(\mathrm{T}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(\mathrm{T}),\mathbb{R}_{>0})$ with a subset of $(\mathbb{R}_{>0})^{d+1}$, the dual map

$$\iota : \Lambda(\mathrm{S}) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(\mathrm{S}),\mathbb{R}_{>0}) \to (\mathbb{R}_{>0})^{d+1}$$

is simply defined by

$$u \mapsto (\langle \lambda_i,u\rangle)_{0\leqslant i\leqslant d}\,.$$

This is an embedding since the representation ρ is faithful.

Lemma 5.1. — *The preimage under ι of the fan $\mathscr{F}$ generated by $\{\mathrm{C}_0,\dots,\mathrm{C}_d\}$ is the fan $\mathscr{F}_\tau$ on $\Lambda(\mathrm{S})$.*

Proof. By definition,

$$\iota^{-1}(\mathrm{C}_i) = \{u \in \Lambda(\mathrm{S})\ ;\ \langle\lambda_i,u\rangle \geqslant \langle\lambda_j,u\rangle, \ \text{ for all j}\} = \{u \in \Lambda(\mathrm{S})\ ;\ \langle\lambda_i-\lambda_j,u\rangle \geqslant 1, \ \text{ for all j}\}.$$

Given a basis Δ of $\Phi(\mathrm{G},\mathrm{S}) \subset \mathrm{X}^*(\mathrm{S})$, we denote by $\mathrm{P}^\Delta_\varnothing$ the corresponding minimal parabolic subgroup of G containing S and by $\lambda_0(\Delta)$ the highest k-weight of ρ with respect to $\mathrm{P}^\Delta_\varnothing$; we also recall that the Weyl cone $\mathfrak{C}(\mathrm{P}^\Delta_\varnothing)$ is defined by the conditions $\alpha \geqslant 1$ for all $\alpha \in \Delta$. If $\lambda_0(\Delta) = \lambda_i$, then $\lambda_i - \lambda_j$ is a linear combination with non-negative coefficients of elements of Δ and thus $\iota^{-1}(\mathrm{C}_i)$ contains $\mathfrak{C}(\mathrm{P}^\Delta_\varnothing)$. Therefore, it follows from the proof of Lemma 2.5 that $\iota^{-1}(\mathrm{C}_i)$ contains the cone

$$\mathrm{C}_\tau(\mathrm{P}^\Delta_\varnothing) = \bigcup_{\substack{\Delta' \\ \lambda_0(\Delta') = \lambda_0(\Delta)}} \mathfrak{C}(\mathrm{P}^{\Delta'}_\varnothing)$$

if λ_i is the highest weight of ρ with respect to $\mathrm{P}^\Delta_\varnothing$.

The inclusion $\mathrm{C}_\tau(\mathrm{P}^\Delta_\varnothing) \subset \iota^{-1}(\mathrm{C}_i)$ is in fact an equality. If it were not, then $\iota^{-1}(\mathrm{C}_i)$ would meet the interior of some Weyl cone $\mathfrak{C}(\mathrm{P}^{\Delta'}_\varnothing)$ with $\lambda_0(\Delta') \neq \lambda_i$. Setting $\lambda_0(\Delta') = \lambda_j$, it would follow that $\iota^{-1}(\mathrm{C}_i \cap \mathrm{C}_j)$ contains a point x of $\mathfrak{C}(\mathrm{P}^{\Delta'}_\varnothing)^\circ$. Such a situation cannot happen: on the one hand, $\iota(x) \in \mathrm{C}_i \cap \mathrm{C}_j$ implies $\lambda_i(x) = \lambda_j(x)$; on the other hand, $\lambda_j - \lambda_i$ is a non-zero linear combination with non-negative coefficients of elements of Δ', hence $\lambda_j - \lambda_i > 1$ on $\mathfrak{C}(\mathrm{P}^{\Delta'}_\varnothing)^\circ$ and thus $\lambda_j(x) > \lambda_i(x)$.

We have therefore $\iota^{-1}(\mathrm{C}_i) = \mathrm{C}_\tau(\mathrm{P}^\Delta_\varnothing)$ if λ_i is the highest k-weight of ρ with respect to $\mathrm{P}^\Delta_\varnothing$, whereas $\iota^{-1}(\mathrm{C}_i^\circ)$ is empty if λ_i doesn't occur among the highest k-weights of ρ. We have checked that the fans $\mathscr{F}_\tau$ and $\iota^{-1}(\mathscr{F})$ have the same cones of maximal dimension; since each face is the intersection of suitable cones of maximal dimension, it follows that $\mathscr{F}_\tau = \iota^{-1}(\mathscr{F})$. □

By the preceding lemma, the affine map $i : \mathrm{A}(\mathrm{S},k) \to \mathrm{A}(\mathrm{T},k)$ can be extended to a continuous injective map

$$i : \overline{\mathrm{A}}_\tau(\mathrm{S},k) \longrightarrow \overline{\mathrm{A}}_\pi(\mathrm{T},k)$$

which is a homeomorphism of $\overline{\mathrm{A}}_\tau(\mathrm{S},k)$ onto the closure of $i(\mathrm{A}(\mathrm{S},k))$ in $\overline{\mathrm{A}}_\pi(\mathrm{T},k)$.

(5.3) As recalled in Section 2, $\overline{\mathscr{B}}_\tau(\mathrm{G},k) \cong \overline{\mathscr{B}}(\mathrm{G},k)_\rho$ can be described as the quotient of $\mathrm{G}(k) \times \overline{\mathrm{A}}_\tau(\mathrm{S},k)$ by the following equivalence relation:

$$(g,x) \sim (h,y) \quad \text{if and only if there exists an element } n \in \mathrm{N}(k) \text{ such that } nx = y \text{ and } g^{-1}hn \in \mathrm{P}_x.$$

Here P_x is defined as $\mathrm{P}_x = \mathrm{N}(k)_x \mathrm{U}_x$, where $\mathrm{N}(k)_x$ is the subgroup of $\mathrm{N}(k)$ fixing x, and where U_x is generated by all filtration steps $\mathrm{U}_\alpha(k)_{-\log\widetilde{\alpha}(x)}$ in the root group $\mathrm{U}_\alpha(k)$, with

$$\widetilde{\alpha}(x) = \sup\{c \in \mathbb{R}_{>0} : x \in \overline{\{\alpha(\cdot - o) \geqslant c\}}.$$

Similarly, $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$ can be described as the quotient of $\mathrm{SL}_\mathrm{V}(k) \times \overline{\mathrm{A}}_\pi(\mathrm{T},k)$ with respect to the analogous equivalence relation involving the stabilizer groups P'_x for $x \in \overline{\mathrm{A}}_\pi(\mathrm{T},k)$.

Composing $i : \overline{\mathrm{A}}_\tau(\mathrm{S},k) \to \overline{\mathrm{A}}_\pi(\mathrm{T},k)$ with the embedding of $\overline{\mathrm{A}}_\pi(\mathrm{T},k)$ in $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$, we obtain a continuous and therefore $\mathrm{N}(k)$-equivariant map $\overline{\rho}_* : \overline{\mathrm{A}}_\tau(\mathrm{S},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$.

Now we want to continue this map to the compactified building $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$.

Lemma 5.2. — *For every $x \in \overline{\mathrm{A}}_\tau(\mathrm{S},k)$ we have $\rho(\mathrm{P}_x) \subset \mathrm{P}'_{i(x)}$, where P_x denotes the stabilizer of x in $\mathrm{G}(k)$ and $\mathrm{P}'_{i(x)}$ denotes the stabilizer of $i(x)$ in $\mathrm{SL}_\mathrm{V}(k)$.*

Proof. If $x \in \mathrm{A}(\mathrm{S},k)$, the claim holds by (5.1), property 2. In general, we have $\mathrm{P}_x = \mathrm{U}(k)_x \mathrm{N}(k)_x$ where $\mathrm{N}(k)_x$ is the stabilizer of x in $\mathrm{N}(k)$. Since $\rho_* : \overline{\mathrm{A}}_\tau(\mathrm{S},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$ is $\mathrm{N}(k)$-equivariant, we find $\rho(\mathrm{N}(k)_x) \subset \mathrm{P}'_{i(x)}$. The group $\mathrm{U}(k)_x$ is generated by all $\mathrm{U}_\alpha(k)_x = \mathrm{U}_\alpha(k)_{-\log\widetilde{\alpha}(x)}$ for $\alpha \in \Phi^{\mathrm{red}}$. Hence it suffices to show $\rho(\mathrm{U}_\alpha(k)_x) \subset \mathrm{P}'_{i(x)}$ for all $\alpha \in \Phi^{\mathrm{red}}$.

If $0 < \widetilde{\alpha}(x) < \infty$, then there exists a sequence (x_n) of points in $\mathrm{A}(\mathrm{S},k)$ converging towards x and such that $\widetilde{\alpha}(x) = \alpha(x_n)$ for all n, hence $\mathrm{U}_\alpha(k)_x = \mathrm{U}_\alpha(k)_{x_n}$ for all n. By (5.1), property 2, it follows that $\rho(\mathrm{U}_\alpha(k)_x) \subset \rho(\mathrm{P}_{x_n})$ is contained in $\mathrm{P}'_{i(x_n)}$. Since $i(x_n)$ converges towards $i(x)$ and $\mathrm{SL}_\mathrm{V}(k)$ acts continuously on $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$, this implies $\rho(\mathrm{U}_\alpha(k)_x) \subset \mathrm{P}'_{i(x)}$.

If $\widetilde{\alpha}(x) = 0$, then $\mathrm{U}_\alpha(k)_x = \{1\}$ and there is nothing to prove.

It remains to address the case where $\widetilde{\alpha}(x) = \infty$, hence $\mathrm{U}_\alpha(k)_x = \mathrm{U}_\alpha(k)$. There exists a sequence (x_n) of points in $\mathrm{A}(\mathrm{S},k)$ converging to x and such that $\lim \alpha(x_n) = \infty$ (observe that x belongs to the closure of each half-space $\{\alpha(\cdot - o) \geq c\}$, with $c \in \mathbb{R}_{\geq 0}$). Any element u of $\mathrm{U}_\alpha(k)$ lies in one of the filtration steps $\mathrm{U}_\alpha(k)_r$; since this filtration is decreasing, u belongs to $\mathrm{U}_\alpha(k)_{x_n}$, hence to the stabilizer P_{x_n}, if n is big enough. By Landvogt's results, this implies that $\rho(u)$ is contained in $\mathrm{P}'_{i(x_n)}$ for n big enough. Since $\mathrm{SL}_\mathrm{V}(k)$ acts continuously on $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$, it follows that $\rho(u)$ is indeed contained in $\mathrm{P}'_{i(x)}$ and the proof is complete. □

It follows immediately from the lemma above that the natural $\mathrm{G}(k)$-equivariant map

$$\mathrm{G}(k) \times \overline{\mathrm{A}}_\tau(\mathrm{S},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k), \quad (g,x) \mapsto \rho(g) \cdot \overline{\rho}_*(x)$$

factors through the equivalence relation defining $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ and thus induces a $\mathrm{G}(k)$-equivariant and continuous map

$$\overline{\rho}_* : \overline{\mathscr{B}}_\tau(\mathrm{G},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$$

extending Landvogt's map ρ_*.

Theorem 5.3. — *The map $\overline{\rho}_* : \overline{\mathscr{B}}_\tau(\mathrm{G},k) \to \overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$ is a $\mathrm{G}(k)$-equivariant homeomorphism of $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ onto the closure of $\rho_*(\mathscr{B}(\mathrm{G},k))$ in $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$.*

Proof. The image of the compact space $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ under $\overline{\rho}_*$ is closed, hence it contains the closure of $\rho_*(\mathscr{B}(\mathrm{G},k))$. On the other hand, any point z in $\rho_*(\overline{\mathscr{B}}(\mathrm{G},k)_\rho)$ is of the form $z=\rho(g)\cdot\overline{\rho}_*(x)$ for some $g\in\mathrm{G}$ and some $x\in\overline{\mathrm{A}}_\tau(\mathrm{S},k)$. If (x_n) is a sequence of points in $\mathrm{A}(\mathrm{S},k)$ converging towards x, then $(\rho(g)\cdot\overline{\rho}_*(x_n))$ is a sequence of points in $\rho_*(\mathscr{B}(\mathrm{G},k))$ converging towards z, hence z is contained in the closure of $\rho_*(\mathscr{B}(\mathrm{G},k))$. Injectivity follows from the fact that any two points of $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ are contained in one compactified apartment by Theorem 1.13 (i).

Therefore, the map $\overline{\rho}_*$ is a continuous bijection between $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ and the closure of $\rho_*(\mathscr{B}(\mathrm{G},k))$ in $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$. Since both spaces are compact, this is a homeomorphism. □

(5.4) We complete this work by identifying the k-rational type τ appearing in Theorems 2.1 and 5.3.

Proposition 5.4. — *The type τ is the unique k-rational type defining the Berkovich compactification $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$. Equivalently, we have*

$$\overline{\mathscr{B}}(\mathrm{G},k)_\rho\cong\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$$

and any Landvogt map $\rho_*:\mathscr{B}(\mathrm{G},k)\to\mathscr{B}(\mathrm{SL}_\mathrm{V},k)$ extends to a $\mathrm{G}(k)$-equivariant homeomorphism between $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$ and a closed subspace of $\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$.

Proof. Applying Theorem 4.8 to the contragredient representation $\check{\rho}$, the Berkovich map $\underline{\check{\rho}}$ provides us with a $\mathrm{G}(k)$-homeomorphism between $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$ and a closed subspace of $\overline{\mathscr{B}}_\delta(\mathrm{SL}_{\mathrm{V}^\vee},k)\cong\overline{\mathscr{B}}_\pi(\mathrm{SL}_\mathrm{V},k)$. Since this map is toral (Proposition 4.9), it satisfies conditions 1 to 3 of (5.1) and we deduce from Theorem 5.3 that the compactifications $\overline{\mathscr{B}}_{t(\rho)}(\mathrm{G},k)$ and $\overline{\mathscr{B}}_\tau(\mathrm{G},k)$ are $\mathrm{G}(k)$-homeomorphic. Thus, τ is the unique k-rational type defining the same Berkovich compactification as the type $t(\rho)$ naturally attached to the absolutely irreducible representation ρ (see [**RTW09**, Appendix C]). □

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